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Materials are intended for a unified and accelerated plane, and solid geometry course and for easy modification and adapation by the experienced or inexperienced teacher. Textbooks to which the guide has been keyed are listed. Units are given for both semesters with time allotments recommended. Each unit is sub-divided into topics and objectives; concepts, vocabulary and symbolism; references; and suggestions. A glossary of geometric terms is included.

Suggestions and supplementary references are intended for use by the geometry teacher in a regular or accelerated class. The guide was written with the unifying concept of mathematical structure in mind. The subject matter should not be taught in the usual segmented fashion. Units include: basic concepts, methods of reasoning, triangles, perpendicular lines and planes-parallel lines and planes, polygons and polyhedrons, inequalities, ratio and proportion, similar polygons, circles and spheres, geometic constructions, locus, coordinate geometry, areas of polygons and circles, geometic solids--areas and volumes and spherical geometry.

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TEACHERS' GUIDE

Ninth Grade Plane and Solid Geometry for the Academically Talented

STATE BOARD OF EDUCATION
OHIO

Prepared by
CLEVELAND PUBLIC SCHOOLS
Division of Mathematics
In Cooperation With
THE OHIO DEPARTMENT OF EDUCATION
Under the Direction of
R. A. HORN
Director, Division of Special Education
Columbus, Ohio
1963

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The Division of Special Education is particularly grateful to the members of the Mathematics Curriculum Committee of the Cleveland Public Schools for their contributions to this publication. The committee was composed of:

Dr. William B. Levenson, Superintendent

Mr. Alva R. Dittrick, Deputy Superintendent

Dr. Harry E. Ritchie, Assistant Superintendent

Mrs. Dorothy E. Norris, Directing Supervisor of Major Work

Mr. Lawrence Hyman, Directing Supervisor of Mathematics

Mr. Martin H. Preuss, Supervisor

Miss E. Jean Thom, Coordinator

Mr. Paul D. Grossberg, Collinwood High School

Mrs. Marlene L. Warburton, Wilbur Wright Junior High School

To each of these people, we offer our sincere thanks and appreciation. We feel that through their efforts a valuable addition has been made in the enrichment and acceleration of junior high school mathematics in Ohio.

THOMAS M. STEPHENS
Administrative Assistant
Division of Special Education

R. A. HORN
Director
Division of Special Education

FOREWORD

During the past three years, the Cleveland Public Schools in cooperation with the Ohio Department of Education have conducted a demonstration project in junior high school mathematics. This demonstration project has been supported by funds that were appropriated through legislative action.

This junior high school mathematics program was designed to augment the horizontal enrichment of the mathematics courses with vertical acceleration of course content. The teachers' guides were developed for the seventh, eighth, and ninth grade mathematics courses by the teachers and supervisors participating in the project.

This teachers' guide is an outgrowth of that demonstration project and is presented to the educators of Ohio as part of our continued efforts to provide for the school children of Ohio. It is my hope that the schools of Ohio will be able to modify or adopt this guide to meet their needs in the area of mathematics.

E. E. HOLT
Superintendent of Public Instruction

INTRODUCTION

Background

In September, 1960, the Cleveland Public Schools in cooperation with the Ohio Department of Education, Division of Special Education began a demonstration project in junior high school mathematics for academically talented students. This mathematics demonstration project was designed to go beyond homogeneous grouping and classroom enrichment. An accelerated program was begun in the seventh grade by combining the seventh and eighth grade programs into one year. Algebra I and II was introduced to these academically talented students in the eighth grade, and a combined plane and solid geometry course was introduced in the ninth grade. By accelerating the academically talented students in the junior high school, the opportunity to take an additional three semesters of college preparatory mathematics would be available to these students in the high school.

After three years, the Cleveland Public Schools have had an opportunity to evaluate the program both objectively and subjectively. The objective evaluation has been done through the use of various standardized tests given to the accelerates, the best regular classes at each grade level, and the regular classes at each grade level. In almost all cases the accelerates scored as well as or better than the groups with which they were compared on the standardized tests. It should be remembered that the accelerated students are at least six months to one and one-half years younger chronologically than the comparison groups in the regular curriculum.

A subjective evaluation was made by questioning administrators, counselors, teachers, and pupils. The general concensus of opinion of these people was that the program has been highly successful and should remain a part of the junior high school curriculum, however it should be sufficiently flexible to meet the changing needs of the school and the pupils involved.

Suggestions for Using the Guide

This teachers' guide contains materials for a unified and accelerated plane and solid geometry course. These materials are presented in such a way that they can be easily adapted and modified to meet the needs of most plane and/or solid geometry classes. The suggestions and supplementary references found in each unit should be a valuable aid to the geometry teacher in a regular or accelerated class.

This guide has been written with the unifying concept of mathematical structure in mind. Therefore, the subject matter cannot be taught in the usual segmented fashion. Because this guide does not follow any one textbook, the geometry teacher should become familiar with the entire guide before attempting to use it. It is also recommended that the teacher review the materials in the accelerated seventh grade Mathematics and eighth grade Algebra courses so that the geometry course content can be integrated into the total program.

It is our hope that this guide can be adapted or modified to meet the needs of both the experienced and inexperienced teacher and thereby lead to the improvement of the secondary school mathematics program.

ARTHUR R. GIBSON
Education Specialist
Programs for the Gifted

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TEXTBOOKS

This guide has been keyed to several textbooks. Note that referral to a particular textbook is designated in the References column by the assigned letter and page.

A. Smith, Rolland R. and Ulrich, James F. Plane Geometry. Yonkers-on-Hudson, New York: World Book Company, 1956.

B. Skolnik, David and Hartley, Miles C. Dynamic Solid Geometry. New York: D. Van Nostrand Company, Inc., 1952.

C. Weeks, Arthur W. and Adkins, Jackson B. A Course in Geometry, Plane and Solid. New York: Ginn and Company, 1961.

D. Goodwin, A. Wilson, Vannatta, Glen D., and Fawcett, Harold P. Geometry, A Unified Course. Columbus, Ohio: Charles E. Merrill Books, Inc., 1961.

E. Morgan, Frank M. and Zartman, Jane. Geometry: Plane · Solid · Coordinate. Boston: Houghton Mifflin Company, 1963.

F. Jurgensen, Ray C., Donnelly, Alfred J., and Dolciani, Mary P. Modern Geometry, Structure and Method. Boston: Houghton Mifflin Company, 1963.

G. Welchons, A. M., Krickenberger, W. R., and Pearson, Helen R. Essentials of Solid Geometry. New York: Ginn and Company, 1959.

SEMESTER I

Unit I	Basic Concepts	13 sessions
Unit II	Methods of Reasoning	16 sessions
Unit III	Triangles	16 sessions
Unit IV	Perpendicular Lines and Planes Parallel Lines and Planes	10 sessions
Unit V	Polygons and Polyhedrons	20 sessions
Unit VI	Inequalities	10 sessions
Unit VII	Ratio and Proportion	5 sessions

SEMESTER II

Unit VIII	Similar Polygons	14 sessions
Unit IX	Circles and Spheres	22 sessions
Unit X	Geometric Constructions	5 sessions
Unit XI	Locus	9 sessions
Unit XII	Coordinate Geometry	10 sessions
Unit XIII	Areas of Polygons and Circles	9 sessions
Unit XIV	Geometric Solids - Areas and Volumes	10 sessions
Unit XV	Spherical Geometry	9 sessions

Two sessions of the second semester are allotted to the administration of the standardized tests.

UNIT I

BASIC CONCEPTS

13 Sessions

Unit I - Basic Concepts (13 sessions)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
SET THEORY To reinforce the concept of set notation	set A set is a collection of objects. $A = \{a, b, c\}$ The symbol $\{\}$ is read, "the set of".
	universal set A universal set is an appropriate set containing all the elements under consideration.
	element (member) Each object in the set is called an element of the set. The symbol $\in$ is read "is an element of". Read $b \in A$ as "b is an element of set A".
	subset Given two sets X and Y, where every element of X is also an element of Y, X is a subset of Y. $X = \{l, m, r\}$ $Y = \{j, k, l, m\}$ The symbol $\subset$ is read "is a subset of". Read $X \subset Y$ as "X is a subset of Y".
	disjoint sets Disjoint sets are sets which have no elements in common. $X = \{l, m, r\}$ $Y = \{n, o, p, q\}$ X and Y are disjoint sets.
	null set (empty set) The set with no elements in it is called the null set (empty set). The null set is usually designated by the symbol $\emptyset$, $\{\}$, or Λ. The symbol $\emptyset$ is preferred. e.g. The set of girls playing baseball for the Cleveland Indians = $\emptyset$..

REFERENCES	SUGGESTIONS
C (5 - 7) E (5 - 8) F (1 - 11)	A brief review of set theory should be sufficient. The past experience of the class should determine the extent of the review necessary.

infinite set An infinite set is a set whose members cannot be counted.

e.g. The set of positive odd integers.

$$A = \{1, 3, 5, 7, . . .\}$$

finite set A finite set is a set whose members can be counted even though the count may be very great.

e.g. The set of even integers between 10 and 16
$B = \{12, 14\}$
e.g. The set of grains of sand on Miami Beach

intersection Given sets A and B where $A = \{3, 6, 9, 12\}$
$B = \{8, 10, 12\}$

$$A \cap B = \{12\}$$

The intersection is the set consisting of all the elements common to both sets.

$A \cap B$ is read "the intersection of A and B".

union Given sets A and B where $A = \{3, 6, 9, 12\}$
$B = \{8, 10, 12\}$
$A \cup B = \{3, 6, 8, 9, 10, 12\}$.

The union of A and B is the set consisting of all the elements of A and all the elements of B.

$A \cup B$ is read "the union of A and B".

REFERENCES	SUGGESTIONS
	1, 3, 5, 7, . . . should be read "1, 3, 5, 7, and so on". __Do__ __not__ use the word "indefinitely" when referring to the infinite. Webster defines __indefinite__ as "undetermined, unmeasured or unmeasurable, though not infinite". He defines __infinite__ as "without limits of any kind, boundless, greater than any assignable quantity of the same kind".

TOPICS AND OBJECTIVES

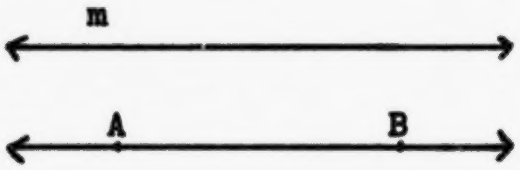
CONCEPTS, VOCABULARY, SYMBOLISM

UNDEFINED TERMS

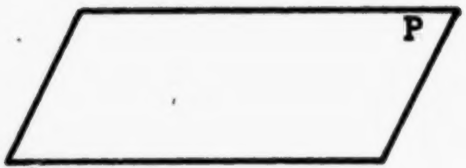
To clarify the undefined terms point, line, plane, space

point A point has no dimensions, only an exact position in space.
A point is usually represented by a dot (the smaller, the better) and is referred to by a capital letter.

line A line is an infinite set of points.
A line has neither width nor depth, only length.
A line is infinite in length, having no end points.
Unless otherwise stated, a line should be interpreted as being a straight line.
A line is usually referred to by a single lower case letter placed near the line or by the letters of two points in the line.



plane A plane is an infinite set of points.
A plane has length and width but no depth.
A plane has the property of being a surface such that a line connecting any two points in its surface lies completely in the surface. This makes the plane infinite because of the line being infinite.
The plane is referred to by a single letter--capital or lower case.
A line is a subset of a plane.



plane

space Space is the set of all possible points.
Space has length, width, and depth.
A plane is a subset of space.

REFERENCES

A	(31 - 32)
C	(6)
D	(15)
E	(4)
F	(22)

SUGGESTIONS

What makes a good definition?
A definition which identifies the word as a member of a set of certain words and distinguishes it from other members of the set.
A definition is reversible.

Example: A pencil is a writing instrument (member of set) using a piece of cylindrical graphite usually encased in wood (distinguishes it from other writing instruments).

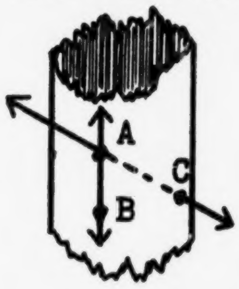
This definition may be reversed as:
A writing instrument using a piece of cylindrical graphite usually encased in wood, is a pencil.

Definitions involving such words as point, line, plane, and space may not be satisfactorily reversed.

Ask the class why a cylindrical surface cannot be considered as a plane.

A cylindrical surface does not meet the requirements of a plane since a line connecting any two points does not necessarily lie wholly in the surface.

Note that in the figure at the right, the line connecting points A and B lies entirely in the surface. However, the line connecting points A and C does not lie in the surface of the cylinder.



A good assignment following a discussion of undefined terms, particularly with reference to the infinite, is to have pupils write a paper on "What I Think Infinity Is". The report need not be restricted to mathematical implications.

Emphasize that points, lines, and planes are abstract images and drawings are merely representations of these abstractions.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
DEFINED TERMS	
To develop a vocabulary of basic concepts	distinct points Consider any point A and any point B different from A. A and B are distinct points.
	betweenness of points Consider A, B, and C as three distinct points in the same line. B is between A and C if AB + BC = AC.
	line segment Line segment AB, designated as $\overline{AB}$, is a subset of a line. A line segment consists of two distinct points, A and B, and all the points between them. A and B are called the end points.
	length of a line segment The length of a line segment is the distance between the end points. Note: The word distance is undefined. Do not use bars or arrows when showing the length of a line segment. e.g. $AB = 5\frac{1}{2}$ inches Two line segments are equal if they have the same length (measure).
	bisect To bisect means to divide into two equal parts.
	midpoint of a line segment B is the midpoint of line segment AC if B lies between A and C such that AB = BC. A midpoint is said to bisect a line segment. Every line segment has one and only one midpoint.

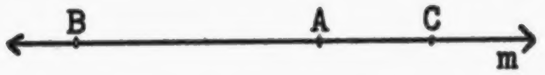
REFERENCES	SUGGESTIONS
C (7 - 8) D (16 - 17) E (10 - 11, 13) F (26 - 30)	It is recommended that pupils keep a notebook for the purpose of relating new vocabulary and symbolism to conceptual material.
	Distance implies the shortest distance between two points which, in a plane, is assumed to be a straight line. In the original Euclid, the concept of distance was undefined. Some present day authors define distance, particularly the distance between two points, as the measure of the line segment joining the two points. Measure is defined as the number of units of a particular unit of measure contained in a line segment. "measure of AB" is written m(AB). Essentially, there is no difference between: the length of AB, written AB and the measure of AB, written m(AB). In this guide, AB will refer to both the segment AB and the measure of AB.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

collinear points A set of points all of which lie in the same line are called collinear points.

half-line Line m is composed of an infinite set of points. Point A in the line divides the line into three disjoint subsets.

Set X = {A}
Set Y = all the points in the half-line in which C lies
Set Z = all the points in the half-line in which B lies

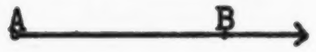


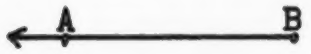
ray Any point in a line separates the line into two half-lines, neither of which includes the given point. The union of the point and one of the half-lines is called a ray.

A is the end point or origin of the ray and B is any other point in the ray.
The ray is referred to by the symbol $\overrightarrow{AB}$.

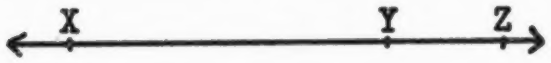
$\overrightarrow{BA}$ refers to the ray whose origin is at B, and A is any other point in the ray.

$\overrightarrow{BA}$ is not the same as $\overrightarrow{AB}$.

ray AB ($\overrightarrow{AB}$) 

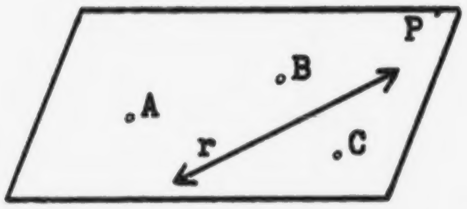
ray BA ($\overrightarrow{BA}$) 

opposite rays Rays YX and YZ are said to be opposite if points X, Y, and Z are collinear and Y lies between X and Z.



REFERENCES	SUGGESTIONS

half-plane A line separates a plane into two half-planes.
A plane, the universal set, is an infinite set of points.
Line r divides the plane into three disjoint subsets.



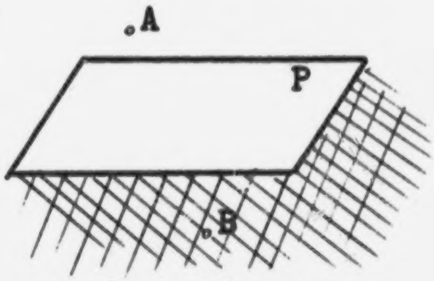
Set L = all the points in the half-plane in which B lies
Set M = all the points in the half-plane in which C lies
Set N = {r}

The line is referred to as the edge of each half-plane.

r divides P into two half-planes.
r is the edge of each half-plane.
A and B lie in the same half-plane and on the same side of r.
B and C lie in different half-planes and on opposite sides of r.

half-space Space is the set of all possible points.
Plane P divides space into three disjoint subsets.

Set E = all the points in the half-space in which A lies
Set F = all the points in the half-space in which B lies
Set G = all the points in P



The plane is called the face of the half-space.

REFERENCES	SUGGESTIONS
	Note that while a line is an edge of infinitely many half-planes, a plane is a face of only two half-spaces.

DETERMINING POINTS, LINES, AND PLANES

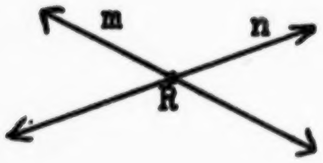
To develop an understanding of the conditions necessary to determine a point, a line, and a plane

determining a point A point is determined by:

1. two intersecting lines.

Set A = all the points in m
Set B = all the points in n

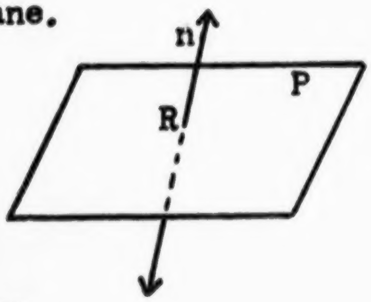
$A \cap B = \{R\}$



2. the intersection of a line and a plane.

Set X = all the points in n
Set Y = all the points in P

$X \cap Y = \{R\}$

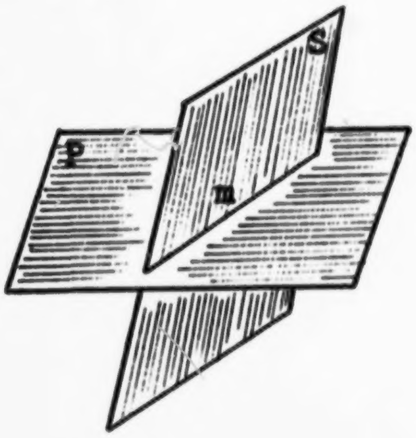


determining a line A line is determined by:

1. two intersecting planes.

Set A = all the points in P
Set B = all the points in S

$A \cap B$ = all the points in m



2. two distinct points.

determining a plane A plane is determined by:

1. three non-collinear points.

2. two intersecting lines.

REFERENCES

A	(89)
B	(13 - 14)
C	(45)
D	(188 - 190)
E	(41)
F	(105 - 106)

SUGGESTIONS

The intersection of any two geometric figures is the set of all the points common to both.

"Determine" means to fix the location of and to limit to a specific number.

Three non-collinear points determine a plane.
This can be demonstrated by having three pupils each hold a pencil with the point up.
Place a book or other flat surface on it.

Why is a stool with three legs always stable but a stool with four legs sometimes not?

> Because three non-collinear points determine one plane, but four non-collinear points determine three planes.

Another way to say "two intersecting lines determine a plane" is "if two lines intersect, one and only one plane contains both these lines."

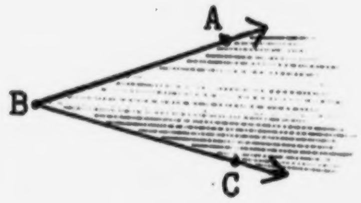
3. a line and a point not in the line.

ANGLES

To develop an understanding of the vocabulary related to angles

angle An angle is the union of two rays with a common end point.

The symbol for angle is ∠.
The symbol for angles is ∠s.



The rays BA and BC are called the sides of angle ABC.

The intersection of $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is the point B, called the vertex of the angle.

Three points in the angle are labeled so that the point at the vertex is listed in the middle.

The interior of ∠ABC is the intersection of the sets of points of the A-side of $\overrightarrow{BC}$ and the C-side of $\overrightarrow{BA}$.

The exterior of ∠ABC is the set of all the points in the plane not in the rays of the angle nor in the interior of the angle.

measure of an angle The measure of an angle depends upon the amount of rotation of a ray about its end point.
The unit of measure used in this course is degree.
A degree is defined as $\frac{1}{360}$ of a complete rotation of a ray about its end point.

1 minute = $\frac{1}{60}$ of a degree; 60 minutes = 1 degree

1 second = $\frac{1}{60}$ of a minute; 60 seconds = 1 minute

REFERENCES

A	(9 - 17, 39 - 45)
C	(22 - 32)
D	(33 - 39)
E	(18 - 23)
F	(30 - 36, 123 - 131)

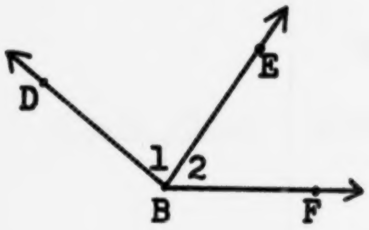
SUGGESTIONS

Let a sheet of notebook paper represent a plane.
Fold the paper in a sharp crease.
  Would this represent two planes which intersect? Yes.
  Is the intersection a straight line? Yes.
  Must two planes intersect each other? No.
  Can they be parallel? Yes.
  Can they be skew? No.

Where there is no chance for confusion, an angle may be denoted by a single letter at its vertex.
However, an angle should never be denoted by one letter where two or more angles have the same vertex.
If the angle in the figure below were called ∠B, it could mean:

∠DBE,
∠EBF, or
∠DBF

∠DBE = ∠1
∠EBF = ∠2



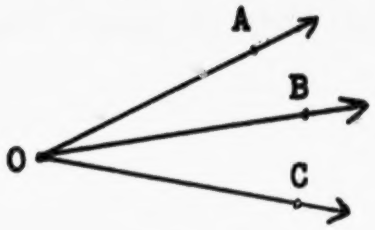
Review the use of the protractor.

Mention radian measure. Pi radians equal 180 degrees.

TOPICS AND OBJECTIVES | **CONCEPTS, VOCABULARY, SYMBOLISM**

betweenness of rays $\overrightarrow{OB}$ is said to lie between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ if:

1. all three rays have a common end point.
2. $\overrightarrow{OB}$ lies so that $\angle AOB + \angle BOC = \angle AOC$.



bisector of an angle $\overrightarrow{OB}$ is said to bisect $\angle AOC$ if:

1. $\overrightarrow{OB}$ lies between $\overrightarrow{OA}$ and $\overrightarrow{OC}$.
2. $\angle AOB = \angle BOC$.

ANGLE CLASSIFICATION

To develop the ability to classify angles according to size

right angle A right angle is formed by a ray making one-fourth of a complete rotation.

Its measure is one-fourth of a complete rotation,
or one-fourth of 360 degrees,
or 90 degrees.

perpendicular lines Perpendicular lines are lines that meet so as to form right angles.

The distance from a point to a line is the length of the perpendicular from the point to the line.

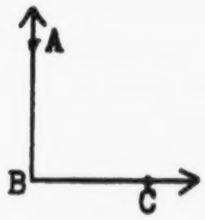
The distance from a point to a plane is the length of the perpendicular from the point to the plane.

straight angle A straight angle is an angle whose sides are opposite rays.

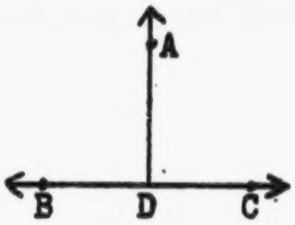
Its measure is one-half of a complete rotation,
or one-half of 360 degrees,
or 180 degrees.

Mention perpendicular lines in conjunction with right angles and again in conjunction with adjacent angles.

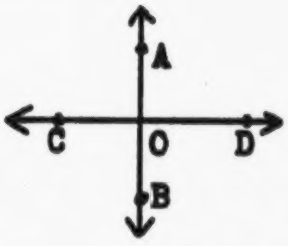
Stress both definitions.
Also demonstrate that the following relations exist:



$\overrightarrow{BA} \perp \overrightarrow{BC}$



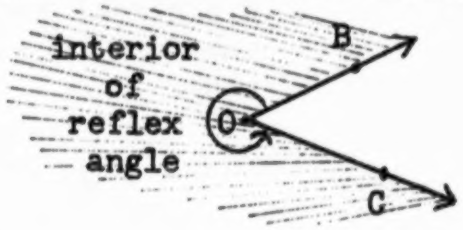
$\overrightarrow{DA} \perp \overleftrightarrow{BC}$



$\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM	
	acute angle	An acute angle is an angle whose measure is > 0 degrees and < 90 degrees.
	obtuse angle	An obtuse angle is an angle whose measure is > 90 degrees and < 180 degrees.
	reflex angle	A reflex angle is an angle whose measure is > 180 degrees.
To develop the ability to classify angles according to their relationships with one another	equal angles	Equal angles are angles whose measures are the same.
	adjacent angles	Adjacent angles are two angles with the same vertex and a common ray between them. Perpendicular lines are two lines that meet to form equal adjacent angles.
	supplementary angles	Supplementary angles are two angles the sum of whose measures is 180°. Each angle is called the supplement of the other. If the exterior sides of two adjacent angles are opposite rays, the angles are supplementary.
	complementary angles	Complementary angles are two angles the sum of whose measures is 90°. Each angle is the complement of the other. Angles need not be adjacent to be complementary or supplementary.

REFERENCES	SUGGESTIONS
	Mention reflex angles. The region which is the interior of a reflex angle can be distinguished by a curved arrow.


interior
of
reflex
angle
O
B
C

TOPICS AND OBJECTIVES

DIHEDRAL ANGLES

To develop an understanding of the concept of dihedral angles as the spatial extension of plane angles

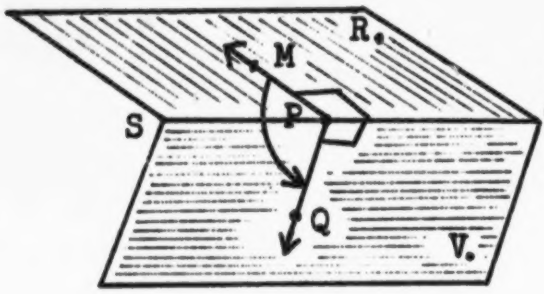
PERSPECTIVE DRAWING

To develop the ability to use perspective in representing figures in space

CONCEPTS, VOCABULARY, SYMBOLISM

dihedral angle A dihedral angle is the union of two half-planes with a common edge.
Each half-plane is called a face of the dihedral angle.
The common edge is called the edge of the dihedral angle.

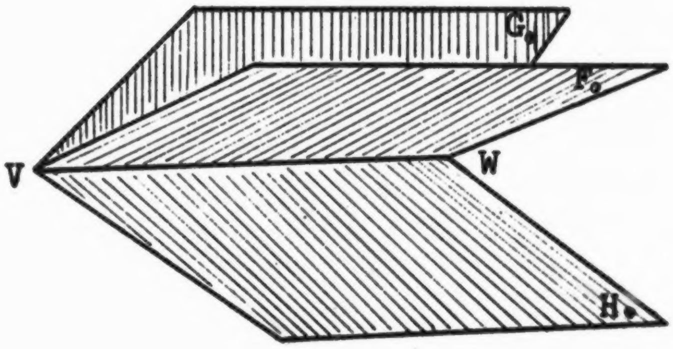
A dihedral angle is named by naming a point in one face, the edge, and then a point in the other face.



Dihedral angle
R-ST-V
or
V-ST-R

plane angle of a dihedral angle The plane angle of a dihedral angle is formed by two rays, one in each face of the dihedral angle, and perpendicular to the edge at the same point.
The measure of the dihedral angle is the same as the measure of the plane angle.

$\angle MPQ$ is the plane angle and the measure of the dihedral angle R-ST-V.

REFERENCES	SUGGESTIONS
B (50 - 51) D (191 - 194) E (79 - 81) F (40)	Is it possible to apply the concepts learned about plane angles to dihedral angles? Dihedral angles can be classified according to size and according to their relationships with one another, using the same classifications as plane angles. The side of a plane angle becomes the face of a dihedral angle. The vertex of a plane angle becomes the edge of a dihedral angle. Betweenness of planes (with regard to dihedral angles) In the figure below, plane F is said to lie between plane G and plane H, if: 1. all three planes have a common edge 2. dihedral angle G-VW-F + dihedral angle F-VW-H = dihedral angle G-VW-H.  Bisector of a dihedral angle Plane F is said to bisect dihedral angle G-VW-H if: 1. F lies between G and H 2. dihedral angle G-VW-F = dihedral angle F-VW-H.
B (5 - 10) E (insert between pp. 184 and 185) F (insert between pp. 32 nd 33. G (8 - 9)	Most pupils have had little, if any, practice in making perspective drawings of solid figures. A day spent in illustrating perspective techniques will be of invaluable aid to future work.

UNIT II

METHODS OF REASONING

16 Sessions

Unit II - Methods of Reasoning (16 sessions)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM	
INDUCTIVE REASONING		
To clarify the nature of inductive reasoning as a method of proof	inductive reasoning	Inductive reasoning is a method of reasoning by which a general conclusion is reached through an examination of a finite set of examples.
To become aware of the strengths and weaknesses of inductive reasoning		
To become aware of the importance of inductive reasoning in the scheme of basic assumptions		

REFERENCES

A (28 - 30)

D (2 - 11)

E (51 - 56)

F (53 - 57, 82 - 83)

SUGGESTIONS

In each of the following experiments, use inductive reasoning to arrive at a conclusion.

1. Draw a triangle.
    a. Measure the angles with a protractor.
    b. Add the results.

2. Follow the same procedure with a quadrilateral.

3. Draw a triangle.
    a. Connect each vertex with the midpoint of the opposite side.

4. Draw a triangle with two equal sides.
    a. Measure the angles opposite those sides.

5. Find the sum of the first n positive integers using the formula $S = \frac{n(n + 1)}{2}$.
    a. Evaluate for S when $n = 4, 5, 6, 8$, and 10.

Use optical illusions to show that things are not always what they appear to be to the eye.

Experiments in inductive reasoning. Are the conclusions justified?

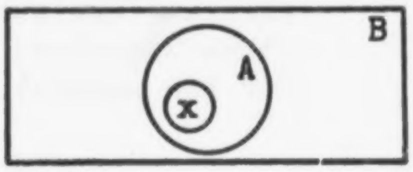
1. Each of six collie dogs Ann has seen has been vicious. Ann concludes all collies are vicious.

2. Mrs. Blake will no longer patronize the corner grocery because last week she bought a bag of potatoes marked ten pounds, but which actually weighed only nine pounds.

3. All the pupils in this geometry class like ice cream. Therefore, all pupils studying geometry like ice cream.

4. Jean's hair has natural-looking deep waves. If you use Jean's shampoo, your hair will be wavy, too.

5. Since the beginning of professional baseball, no team has ever won the pennant without at least one .300 hitter. Since Cleveland has no .300 hitters, the team has no chance for the pennant. (Assume Cleveland has no .300 hitters.)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
DEDUCTIVE REASONING	
To clarify deductive reasoning as a method of proof	deductive reasoning: Deductive reasoning is the method of reasoning by which conclusions are arrived at from accepted statements.
To relate deductive reasoning to the traditional methods of geometric proof	syllogism: A syllogism is an argument made up of three statements: 1. Major premise - an accepted general statement 2. Minor premise - a specific or particular statement 3. Conclusion Or, in set notation: 1. $A \subset B$ 2. $x \in A$ 3. $x \in B$ Or, represented in a Venn diagram:  Essentially Venn diagrams are the same as Euler's circles.

REFERENCES	SUGGESTIONS
	6. It has rained on every Halloween day for the past four years. This year, it certainly will rain on that day. 7. Every time 2 + 2 has been added, since the dawning of creation, the sum has been 4. It is, therefore, an indisputable fact. While it cannot be definitely stated which of the above conclusions are justified and which are not, it seems prudent to say that the justification for any conclusion is directly proportional to the number of examples examined. There are no foolproof rules for induction. Especially important to the understanding of inductive reasoning is example No. 7. 2 + 2 = 4 and other similar basic "facts" are inductively arrived at through countless trials. *Inductive assumptions such as these are the basis for deductive reasoning*. Because of the inherent weakness of inductive reasoning, we cannot be *absolutely* sure of any conclusion reached inductively. For practicality as well as for convenience, we accept such "facts".
A (196 - 198) D (51 - 59) E (56 - 59) F (87 - 90) Shute, W. G., Shirk, W. W., and Porter, G. F. *Supplement to Plane Geometry*, (34-43)	
	The construction of a Venn diagram for each syllogism will aid in the recognition and prevention of invalid reasoning.

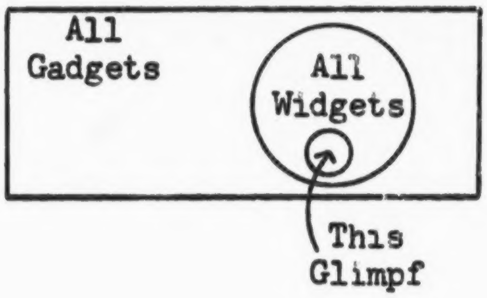
TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

Examples of syllogisms: (1. Major premise 2. Minor premise 3. Conclusion)

A. 1. All widgets are gadgets.
   2. This glimpf is a widget.
   3. This glimpf is a gadget.

In set notation
   1. widgets $\subset$ gadgets
   2. this glimpf $\in$ widgets
   3. this glimpf $\in$ gadgets

In a Venn diagram:



B. 1. All academically talented pupils study combined plane and solid geometry.
   2. We are academically talented pupils.
   3. We are studying combined plane and solid geometry.

C. 1. All composite numbers can be factored.
   2. 3,893,630 is a composite number.
   3. 3,893,630 is factorable.

Have pupils bring in examples of syllogisms. Another possibility is to make syllogisms with one of the three statements missing and require the class to supply the missing statement.

A. 1. _ _ _ _ _ _ _ _ _ _ _
   2. Mr. Beasley is a mailman.
   3. Mr. Beasley has sore feet.

B. 1. All medicines on this shelf are poison.
   2. _ _ _ _ _ _ _ _ _ _ _
   3. This bottle of medicine is poison.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

The deductive process used to arrive at the conclusion of a syllogism is either valid or invalid.

If both the major and minor premise are assumed to be true, a true conclusion will result from valid reasoning and a false conclusion will result from invalid reasoning.

If either the major or minor premise is false or if both are false, valid reasoning could result in either a true or false conclusion!

if-then statements	Deductive reasoning is the process of drawing valid conclusions from accepted statements. Every statement that we prove can be stated in the "if-then" form.
hypothesis (Hyp.)	The clause following the word "if" of a statement in the "if-then" form is called the hypothesis of the statement.
conclusion (Con.)	The clause following the word "then" of a statement in the "if-then" form is called the conclusion of the statement.

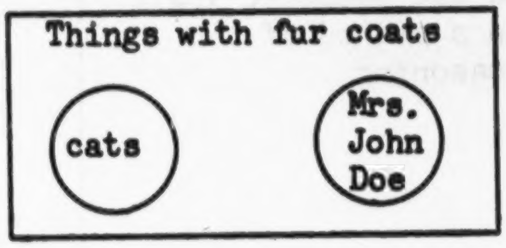
REFERENCES

SUGGESTIONS

Example of an invalid syllogism:

1. Major premise
2. Minor premise
3. Conclusion

1. All cats have fur coats
2. Mrs. John Doe has a fur coat.
3. Mrs. John Doe is a cat.



Both premises are true, but the minor premise does not classify "Mrs. John Doe" as an element of the set of cats. Therefore, the conclusion is false.

Example of a valid syllogism with a false conclusion:

1. All two-legged creatures are human beings.
2. My canary, Tweety Pie, has two legs.
3. My canary, Tweety Pie, is a human being.

The reasoning is correct, and the minor premise relates correctly to the major premise, but the major premise is false. Therefore, the conclusion is false.

Example of a valid syllogism with a true conclusion, but with one or more false premises:

1. All farmers are residents of the United States.
2. All Ohioans are farmers.
3. All Ohioans are residents of the United States.

The conclusion is true and the reasoning is valid, but both the major and minor premises are false.

A (53 - 62)
C (35 - 38)
D (59 - 61)
F (91 - 94)

Example: An obtuse angle is greater than an acute angle.

Rewritten in "if-then" form:
If an angle is obtuse, then it is greater than an acute angle.

Hyp. An angle is obtuse.
Con. The angle is greater than an acute angle.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM	
IMPLICATION To develop the ability to use symbolic logic as a method of reasoning	**implication**	All statements written in the "if-then" form can be symbolized. Let H represent the hypothesis. Let C represent the conclusion. $H \longrightarrow C$ The arrow ($\longrightarrow$) means "if H then C" but is read "H implies C".
	converse	The converse of the statement $H \longrightarrow C$ is $C \longrightarrow H$. The converse of an implication is not necessarily true.
	inverse	The inverse of the statement $H \longrightarrow C$ is "not $H \longrightarrow$ not C". In the notation of symbolic logic, this is $\sim H \longrightarrow \sim C$. The inverse of an implication is not necessarily true.
	contrapositive	The contrapositive of the statement $H \longrightarrow C$ is "not $C \longrightarrow$ not H". ($\sim C \longrightarrow \sim H$). The contrapositive is sometimes called the inverse of the converse. If an implication is true, then the contrapositive is always true.
ASSUMPTIONS To become acquainted with the axioms and postulates needed in elementary deductive reasoning	**proposition**	A proposition is a general statement concerning relationships.
	postulate	A postulate is a geometric proposition accepted without proof. It is an assumption.

REFERENCES

A (124 - 128, 282 - 284)

C (74 - 76)

D (62 - 66)

E (148 - 149, 162 - 164)

Meserve, B. E. and Sobel, M. A. Mathematics for Secondary School Teachers (192-220)

Brumfiel, C. F., Eicholz, R. E., and Shanks, M. E. Geometry (21-42)

SUGGESTIONS

Example showing implication:
Let H stand for "Mary has a toothache."
Let C stand for "Mary visits the dentist."

Then $H \longrightarrow C$ means "If H then C," or "If Mary has a toothache, then Mary visits the dentist."

Implication: If Mary has a toothache, then Mary visits the dentist.
Converse: If Mary visits the dentist, then Mary has a toothache. (Not necessarily true as Mary may visit the dentist for a regular checkup.)
Inverse: If Mary does not have a toothache, then Mary does not visit the dentist. (Not necessarily true as Mary may visit the dentist for other reasons.)
Contrapositive: If Mary does not visit the dentist, then Mary does not have a toothache. (True if implication is true.)

Examples of Implication, Converse, Inverse, and Contrapositive:

1. $H \longrightarrow C$: If the sun shines, I am in a good mood.
   $C \longrightarrow H$: If I am in a good mood, the sun is shining.
   $\sim H \longrightarrow \sim C$: If the sun is not shining, I am not in a good mood.
   $\sim C \longrightarrow \sim H$: If I am not in a good mood, the sun is not shining.

2. $H \longrightarrow C$: If $x + 3 = 9$, then $x = 6$.
   $C \longrightarrow H$: If $x = 6$, then $x + 3 = 9$ (converse is true here)
   $\sim H \longrightarrow \sim C$: If $x + 3 \neq 9$, then $x \neq 6$. (inverse is true here)
   $\sim C \longrightarrow \sim H$: If $x \neq 6$, then $x + 3 \neq 9$.

3. Have pupils make up their own implications and complete the converses, inverses, and contrapositives.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	axiom An axiom is a proposition, general in nature, accepted without proof. It is also an assumption.

1.	Identity axiom	(reflexive axiom) Any quantity is equal to itself.
2.	Symmetry axiom	An equality may be reversed.
3.	Transitive axiom	Two quantities equal to the same or equal quantities are equal to each other.
4.	Substitution axiom	A quantity may be substituted for its equal in any expression without changing the value of the expression.
5.	Addition axiom	If equal quantities are added to equal quantities, the sums are equal.
6.	Subtraction axiom	If equal quantities are subtracted from equal quantities, the differences are equal.
7.	Multiplication axiom	If equal quantities are multiplied by equal quantities, the products are equal. Special case: Doubles of equals are equal.
8.	Division axiom	If equal quantities are divided by equal non-zero quantities, the quotients are equal. Special case: Halves of equals are equal.
9.	Powers axiom	Equal powers of equal quantities are equal.
10.	Roots axiom	The absolute value of equal roots of equal positive quantities are equal.
11.	Axiom of the whole	The whole of any quantity is equal to the sum of all of its parts.

REFERENCES	
A	(84)
C	(39 - 41)
D	(70 - 74)
E	(60 - 61)
F	(100 - 101)

SUGGESTIONS

Examples:

1. $a = a$; $AB = AB$; $\angle F = \angle F$

2. If $a = b$, then $b = a$. If $AB = CD$, then $CD = AB$.

3. If $a = b$, and $a = c$, then $b = c$.
   If $a = b$, and $c = d$, and $a = c$, then $b = d$.

4. If $a = b + c$ and $b = 5$, then $a = 5 + c$.

5. If $AB = CD$ and $EF = GH$, then $AB + EF = CD + GH$.
   CAUTION: $AB + CD \neq EF + GH$.

6. If $AB = CD$ and $EF = GH$ then $AB - EF = CD - GH$.
   CAUTION: $AB - CD \neq EF - GH$.

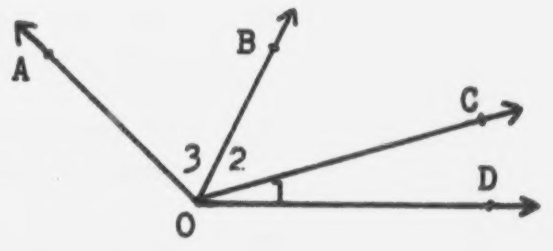
7. If $\angle A = \angle B$, then $5(\angle A) = 5(\angle B)$.

8. If $AB = CD$, then $\frac{AB}{3} = \frac{CD}{3}$.

9. If $AB = CD$, then $(AB)^2 = (CD)^2$.

10. If $x^2 = y^3$, then $\left|\sqrt{x^2}\right| = \left|\sqrt{y^3}\right|$.

11. In the figure, $\angle 1 + \angle 2 + \angle 3 = \angle DOA$.



TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
POSTULATES	
To become acquainted with some elementary postulates	1. A line segment has one and only one midpoint. 2. The shortest distance between two points is a straight line. 3. An angle has only one bisector. 4. A point is determined by the intersection of two lines or the intersection of a line and a plane. 5. A line is determined by two distinct points or by two intersecting planes. 6. A plane is determined by three non-collinear points, by two intersecting lines, or by a line and a point not in the line. 7. All right angles are equal. 8. All straight angles are equal. 9. The shortest distance from a point to a line is the perpendicular from the point to the line. 10. If two points lie in a plane, then a line connecting the two points lies in the plane.
PROVING BASIC THEOREMS	
To develop the ability to prove certain fundamental theorems using axioms and postulates	theorem (Th.) A theorem is a proposition that is proved by deductive reasoning. Th. If two angles are supplementary to the same or equal angles, they are equal. Th. If two angles are complementary to the same or equal angles, they are equal. Th. Vertical angles are equal. Th. If two dihedral angles are supplementary to the same or equal dihedral angles, they are equal. Th. If two dihedral angles are complementary to the same or equal dihedral angles, they are equal. Th. If planes intersect, the vertical dihedral angles are equal. Symbol for "therefore" is ∴.

As a good enrichment problem, have pupils find the fallacy in the following:

$$\begin{array}{rcrl} 2\text{ lb.} & = & 32\text{ oz.} & \text{Given}\\ \frac{1}{2}\text{ lb.} & = & 8\text{ oz.} & \text{Division axiom}\\ \hline 1\text{ lb.} & = & 256\text{ oz.} & \text{Multiplication axiom}\end{array}$$

A (74, 84, 89)

C (41 - 46)

D (26)

F (104, 108)

A (47 - 50, 78 - 80)

Introduce formal proof.

Example: Th. If two angles are supplementary (supp.) to the same angle or equal angles, they are equal.

Hyp. $\angle A$ is supp. to $\angle C$.
$\angle B$ is supp. to $\angle C$.

Con. $\angle A = \angle B$

Statements	Reasons
1. $\angle A$ is supp. $\angle C$ $\angle B$ is supp. $\angle C$	1. Given
2. $\angle A + \angle C = 180°$ $\angle B + \angle C = 180°$	2. Definition of supplementary angles
3. $\angle C = \angle C$	3. Identity axiom
4. $\angle A = 180° - \angle C$ $\angle B = 180° - \angle C$	4. Subtraction axiom
5. $\therefore \angle A = \angle B$	5. Transitive axiom

UNIT III

TRIANGLES

16 Sessions

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

TRIANGLE

To become familiar with the triangle and its parts

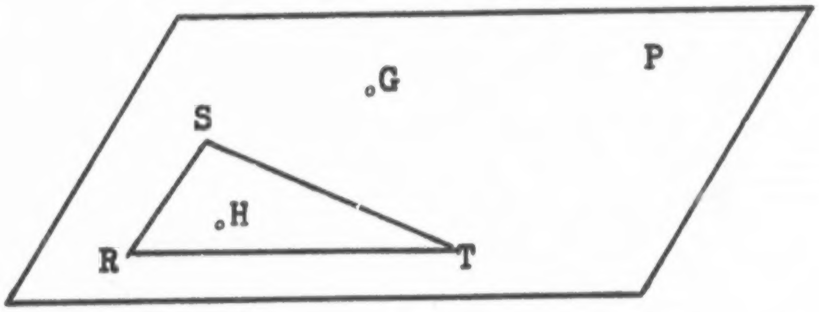
triangle The union of three line segments joining any three non-collinear points is a triangle.

A triangle separates any plane into three disjoint subsets.

Set A = all the points in the triangle (in the three line segments)

Set B = all the points in the interior of the triangle

Set C = all the points in the exterior of the triangle



S ∈ Set A

The interior of a triangle is the intersection of the sets of points in

the interior of ∠SRT and the interior of ∠STR, or
the interior of ∠STR and the interior of ∠TSR, or
the interior of ∠TSR and the interior of ∠SRT.

H ∈ Set B

The exterior of a triangle is the set of all the points not in the triangle nor in the interior of the triangle.

G ∈ Set C

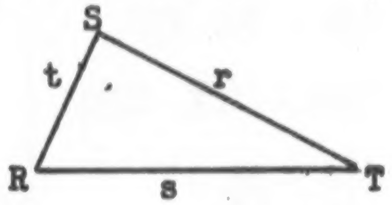
REFERENCES	SUGGESTIONS
A (2, 10, 23, 289) C (58 - 59, 91) D (101 - 102, 107) E (36 - 39, 89 - 90)	

TOPICS AND OBJECTIVES | CONCEPTS, VOCABULARY, SYMBOLISM

vertex (pl. vertices) In the figure, each of the three points R, S, T is a vertex of the triangle.

A triangle is named by naming the vertices.

The symbol for triangle is "△".



Triangle RST or
△ RST

The line segments are called the sides of the triangle.

Side r is opposite $\angle R$. $\angle R$ is opposite side r and included between sides s and t.
Side s is opposite $\angle S$. $\angle S$ is opposite side s and included between sides r and t.
Side t is opposite $\angle T$. $\angle T$ is opposite side t and included between sides r and s.

TRIANGLES CLASSIFIED BY SIDES

To develop the ability to classify triangles according to their sides

scalene A scalene triangle is a triangle with no equal sides.

isosceles An isosceles triangle is a triangle with two equal sides.
The equal sides are called legs.
The angle formed by the legs is called the vertex angle.
The side opposite the vertex angle is called the base.
The two angles adjacent to the base are called the base angles.

equilateral An equilateral triangle is a triangle with three equal sides.

The set of isosceles triangles is a subset of the set of equilateral triangles. The converse of this statement is not true.

TRIANGLES CLASSIFIED BY ANGLES

To develop the ability to classify triangles according to their angles

acute An acute triangle is a triangle with three acute angles.

obtuse An obtuse triangle is a triangle with one obtuse angle.

REFERENCES	SUGGESTIONS
	When drawing a triangle, a scalene triangle should be drawn unless another type is specifically indicated. The scalene triangle is considered the general form of a triangle. Furthermore, the scalene triangle drawn should not contain a right angle if the triangle is to be considered general in nature.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	right — A right triangle is a triangle with one right angle. The side opposite the right angle is called the hypotenuse. The sides adjacent to the right angle are called the legs.
	equiangular — An equiangular triangle is a triangle with three equal angles.
	The set of equiangular triangles is equal to the set of equilateral triangles. The converse of this statement is true.
LINE SEGMENTS AND ANGLES ASSOCIATED WITH TRIANGLES	
To become familiar with special line segments and angles associated with triangles	perimeter — The perimeter of a triangle is the sum of the lengths of the three sides.
	altitude — An altitude of a triangle is the line segment drawn from a vertex perpendicular to the opposite side or side produced.
	median — A median of a triangle is the line segment drawn from a vertex to the midpoint of the opposite side.
	base — The base of a triangle may be any side. The base is generally the side upon which the altitude is constructed or is thought to be constructed. In an isosceles triangle, the base is the side opposite the vertex angle.
TRIANGLE CONGRUENCY	
To develop the concept of triangle congruency	congruent triangles — Congruent triangles are triangles whose corresponding parts, angles and sides, are equal.
	Congruent polygons are polygons whose corresponding parts, angles and sides, are equal. Congruent triangles are a subset of the universal set of congruent polygons.
	The symbol for the phrase "is congruent to" is ≅.

REFERENCES	SUGGESTIONS
A (64 - 98) C (60 - 61) D (116 - 123) E (91 - 104) F (189 - 203)	According to Euclid, congruent figures are figures which can be superimposed so that they coincide. This definition depends on undefined concepts such as motion, the ability to keep an object rigid during motion, and the ability to move an object to a desired place. In this guide, the procedure is to take congruence as an undefined concept.

Methods of proving triangles congruent are:

If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent.

This is customarily abbreviated s.a.s. = s.a.s.

If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, they are congruent.

This is customarily abbreviated a.s.a. = a.s.a.

If two triangles have three sides of one equal respectively to three sides of the other, they are congruent.

This is customarily abbreviated s.s.s. = s.s.s.

The above three methods are postulated and are to be added to the postulates previously listed.

Corresponding parts of congruent triangles are equal.

This is customarily abbreviated as C.p.c.t.e.

The unit on construction will not be taken until the second semester. It is sufficient at this time to make triangles congruent using a protractor and a rule instead of the compass and straightedge.

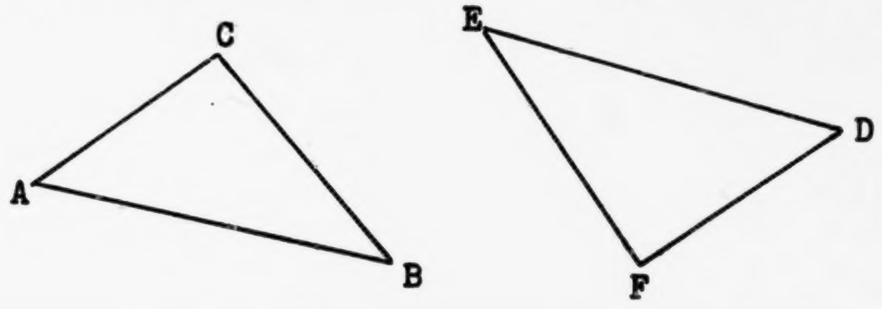
As an experiment in showing s.a.s. = s.a.s., have the pupils draw two triangles each with sides of 2" and 3" and an included angle of 40°. Cut out one triangle and see if it fits or coincides with the other.

Similar experiments can be performed showing a.s.a. = a.s.a. and s.s.s. = s.s.s.

Another method is to have each pupil draw two triangles to his own specifications to see if they coincide. Continue these experiments until the pupils inductively conclude that the triangles are congruent.

In listing the corresponding parts of congruent triangles, the order is of utmost importance. Stating that $\triangle ABC \cong \triangle DEF$ implies:

$\angle A = \angle D$
$\angle B = \angle E$
$\angle C = \angle F$
AC = DF
AB = DE
BC = EF

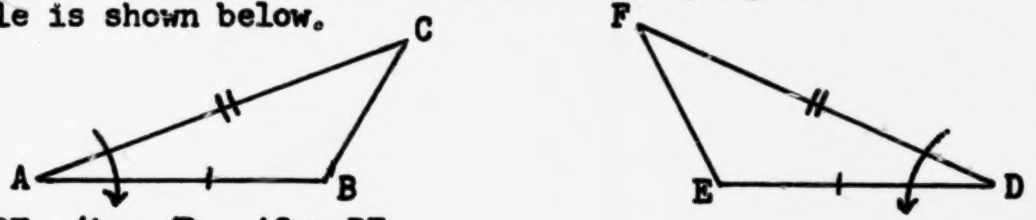


In the above drawing, it would be incorrect to state that $\triangle ABC \cong \triangle EFD$.

If in the above triangles, $\angle A = \angle D$, then the side opposite $\angle A$ is equal to the side opposite $\angle D$.

Some congruent triangles are overlapping parts of geometric figures and are confusing to the eye. When dealing with overlapping triangles, it is suggested that the figures be "pulled apart" and that colored chalk and colored pencils be used to identify the corresponding parts.

It is an aid to pupils to mark corresponding parts of congruent triangles with any of a variety of identifying marks. An example is shown below.



AB = DE, $\angle A = \angle D$, AC = DF

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PROVING THEOREMS USING CONGRUENT TRIANGLES	
To develop the ability to prove some theorems by means of congruent triangles	Th. (prove formally) If two sides of a triangle are equal, the angles opposite those sides are equal. (This theorem may also be stated: The base angles of an isosceles triangle are equal.) Th. (prove formally) If two angles of a triangle are equal, the sides opposite those angles are equal. (This theorem may also be stated: If two angles of a triangle are equal, the triangle is isosceles.)

REFERENCES	
A	(103 - 110, 112 - 113)
C	(71 - 84)
D	(135 - 142)
E	(104 - 117)
F	(220, 222 - 225)

SUGGESTIONS

Because of the restrictions of time, it is suggested that a limited number of "key" theorems throughout the course be proved formally. The rest should be postulated. Those theorems that should be proved formally are so marked. In the absence of any directions, the theorem may be postulated.

Pupils should be taught to outline a plan of action before writing a formal proof.

The first two theorems are proved formally for two reasons:

1. To give the pupil much-needed practice in formal proof
2. To give examples of converse theorems. The converse of any implication is not necessarily true. It is wise to reinforce this concept and convey the idea that because a theorem is proved true, the converse does not automatically follow.

The policy of using specific notation $\overleftrightarrow{AB}$, $\overline{AB}$, and $\overrightarrow{AB}$ has been initiated in order to discipline the pupils in exact thinking. However, it is a more common practice to simply use AB where no ambiguity can arise. That is, AB may refer to $\overleftrightarrow{AB}$, $\overline{AB}$, or $\overrightarrow{AB}$.

From time to time, the pupils should be asked to determine whether a certain notation denotes a line, line segment, a ray, or the measure of a line segment.

Similarly, the symbol $\angle XYZ$ can be used to name an angle or to denote its measure. The context indicates the meaning.

As soon as a theorem has been proved or postulated, it may be used as an acceptable reason in proving other theorems.

auxiliary line An auxiliary line is an extra line not given in the hypothesis of a problem. This line is drawn to help with the proof. The line is usually broken to distinguish it from the given lines.

corollary A corollary is a theorem that is easily proved by the use of a previous theorem. The abbreviation is Corol.

Corol. An equilateral triangle is also equiangular, and conversely.

Corol. The medians to the legs of an isosceles triangle are equal.

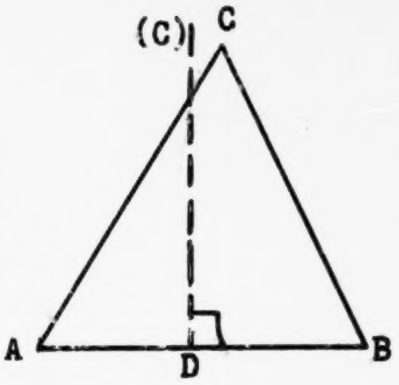
Th. The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

Th. A line segment that connects the vertex angle of an isosceles triangle with the midpoint of the base bisects the vertex angle and is perpendicular to the base.

Th. If line segments are drawn from any point on the perpendicular bisector of a line segment to the ends of the line segment, they are equal.

Th. (prove formally) If two points are each equally distant from the ends of a line segment, a line connecting the two points is the perpendicular bisector of the segment.

Postulate: If two right triangles have the hypotense and an acute angle of one equal to the hypotenuse and an acute angle of the other, they are congruent. This may be abbreviated as "rt. △s h.a. = h.a." This postulate should now be added to the list of postulates.

REFERENCES	SUGGESTIONS
	A common error is to place too many conditions on an auxiliary line. Example: Draw CD as the ⊥ bisector of AB in △ ABC. CD may or may not pass through C. CD can be the median to AB or the perpendicular from C to AB, but not necessarily both. 
	The corollary about the medians to the legs of an isosceles triangle is an example of overlapping triangles.
	To determine which postulate to use in proving two triangles congruent, first discover if there is one pair or if there are two or three pairs of equal corresponding sides. If there is one pair, use a.s.a. = a.s.a. If there are two pairs, use s.a.s. = s.a.s. If there are three pairs, use s.s.s. = s.s.s
A (251) Report of the Commission on Mathematics, Appendices. (166-168)	Introduce the fallacy, "Every triangle is isosceles". This will reinforce such concepts as betweenness of points and points lying inside or outside of a triangle. Note the two references at the left.
A (114-116, 130-131, 134-138)	For superior students and as a challenge for all the class, "Problems for Pacemakers" found in reference A are highly recommended.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. If two right triangles have the hypotenuse and a leg of one equal to the hypotenuse and a leg of the other, they are congruent. This theorem may be abbreviated as "rt. △s h.l. = h.l." Th. If the legs of one right triangle are equal to the legs of another right triangle, the triangles are congruent. This theorem may be abbreviated as "rt. △s l.l. = l.l." Th. If lines are drawn from any point in the bisector of an angle perpendicular to the sides of the angle and terminated by the sides, they are equal.
To develop the ability to use congruent triangles in proving theorems in solid geometry	a line perpendicular to a plane A line is perpendicular to a plane if it is perpendicular to every line in the plane through its foot. The foot is the point of intersection of the line and the plane.

REFERENCES

A (132 - 133)

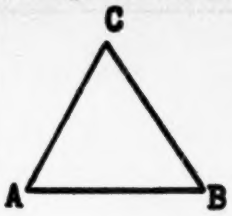
B (22 - 23)

C (101 - 107)

D (141 - 147)

SUGGESTIONS

The following unusual proof will be of interest to students as it was created by an IBM 704 electronic digital computer.



Given: $\triangle$ ABC, with AC = BC
To prove: $\angle A = \angle B$
Plan: Prove $\triangle$ ACB $\cong$ $\triangle$ BCA

Proof

Statements	Reasons
1. AC = BC	1. Given
2. BC = AC	2. Symmetry axiom
3. AB = BA	3. Identity axiom
4. $\triangle$ ACB $\cong$ $\triangle$ BCA	4. s.s.s. = s.s.s.
5. $\therefore \angle A = \angle B$	5. C.p.c.t.e.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. (prove formally as an exercise) If a line is perpendicular to each of two intersecting lines in a plane at their point of intersection, it is perpendicular to the plane of these lines.
	Th. (prove formally as an exercise) If from a point in a perpendicular to a plane, line segments are drawn oblique to the plane meeting the plane at equal distances from the foot of the perpendicular, the segments are equal.
	Th. (converse to the above theorem--prove formally as an exercise) If from a point in the perpendicular to a plane equal line segments are drawn oblique to the plane, their distances from the foot of the perpendicular are equal.

REFERENCES

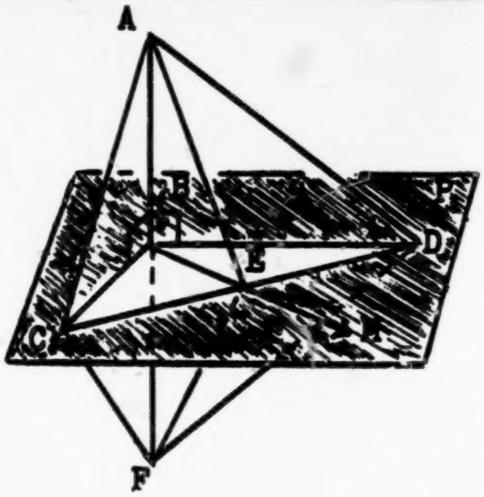
B (29)
D (145)
G (11)

SUGGESTIONS

Given: $\overline{AB} \perp \overline{BC}$ and $\overline{BD}$ in plane P

To prove: $\overline{AB} \perp P$

Plan: Through B draw any line BM in P. Prove $AB \perp BM$. Draw CD intersecting BM at E. Extend AB to F so that $AB = BF$. Draw AC, AD, AE; FC, FD, FE.



Proof

Statements	Reasons
1. $AC = CF$, $AD = DF$	1. If two line segments are drawn from the perpendicular bisector of a line segment to the ends of the line segment, they are equal.
2. $CD = CD$	2. Identity axiom
3. $\triangle ACD \cong \triangle FCD$	3. s.s.s. = s.s.s.
4. $\angle ACE = \angle FCE$	4. C.p.c.t.e.
5. $CE = CE$	5. Identity axiom
6. $\triangle ACE \cong \triangle FCE$	6. s.a.s. = s.a.s.
7. $AE = FE$	7. C.p.c.t.e.
8. $AB = BF$	8. Construction
9. $AB \perp BM$	9. If two points are each equally distant from the ends of a line segment, they determine the perpendicular bisector of the line segment.
10. $\therefore AB \perp P$	10. Definition of a perpendicular to a plane.

UNIT IV

PERPENDICULAR LINES AND PLANES

PARALLEL LINES AND PLANES

10 Sessions

Unit IV - Perpendicular Lines and Planes, Parallel Lines and Planes (10 sessions)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PERPENDICULAR LINES AND PLANES	
To develop an understanding of the concepts of perpendicular lines and planes	Postulate: In a given plane containing a given line, there is one and only one line perpendicular to the given line at any point in the given line.
	Th. Given any line and any point not in the line, there is one and only one line through the given point and perpendicular to the given line.
	Th. Given any plane and any point not in the plane, there is one and only one line through the given point and perpendicular to the given plane.
	perpendicular planes If two planes intersect so that the adjacent dihedral angles are equal, the planes are perpendicular.

REFERENCES

B (27)

C (97 - 107)

D (194)

G (9 - 13)

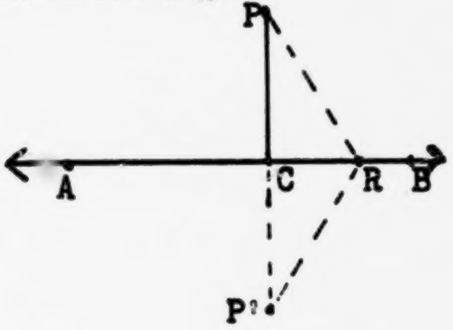
SUGGESTIONS

This theorem may be proved formally as an exercise.

Given: AB with point P not in AB. PC $\perp$ AB at C. R is any point in AB distinct from C.

To prove: PR not $\perp$ AB.

Plan: Extend PC to P' so that PC = P'C. Draw PR and P'R.

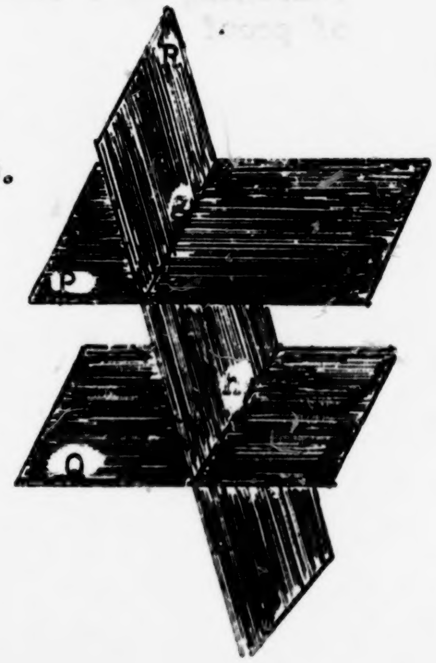


Proof

Statements	Reasons
1. PC = P'C	1. So drawn
2. PP' $\perp$ AB	2. Given
3. $\angle$PCR = $\angle$P'CR	3. Perpendicular lines form equal adjacent angles.
4. CR = CR	4. Identity axiom
5. $\triangle$PCR $\cong$ $\triangle$ P'CR	5. s.a.s. = s.a.s.
6. $\angle$PRC = $\angle$P'RC	6. C.p.c.t.e.
7. $\angle$PRC + $\angle$P'RC = $\angle$PRP'	7. Axiom of the whole
8. 2($\angle$PRC) = $\angle$PRP'	8. Substitution axiom
9. $\angle$PRC = $\frac{1}{2}$($\angle$PRP')	9. Division axiom
10. PCP' is a straight line.	10. Given
11. PRP' is not a straight line, and $\angle$PRP' is not a straight angle.	11. Two points determine only one straight line.
12. $\angle$PRC is not a right angle.	12. A right angle is one-half a straight angle.
13. PR is not $\perp$ AB.	13. Definition of perpendicularity

The pupil, when formally proving theorems as an exercise, should be periodically required to give a complete statement in sentence form as a reason.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PARALLEL LINES AND PLANES To develop an understanding of the concepts of parallel lines and planes	**parallel lines** Parallel lines are two lines in the same plane that do not meet. The symbol for parallel is " $\parallel$ ". The symbol for not parallel is " $\nparallel$ ". **skew lines** Skew lines are two lines that do not lie in any one plane. **parallel planes** Parallel planes are planes that have no point in common. **a line parallel to a plane** A line is parallel to a plane if the line and the plane have no point in common. Th. If two parallel planes are cut by a third plane, the lines of intersection are parallel.



This theorem may be proved as an exercise.

Given: Plane P || plane Q.
P and Q cut by plane R
in lines g and h.

To prove: g || h

Plan: Show that lines g and h satisfy the definition of parallel lines.

Proof

Statements	Reasons
1. g is a straight line and h is a straight line.	1. Two intersecting planes determine a straight line.
2. Lines g and h lie in one plane.	2. Given
3. g, which lies in P, and h, which lies in Q, have no point in common.	3. Definition of parallel planes
4. ∴ g || h.	4. Definition of parallel lines.

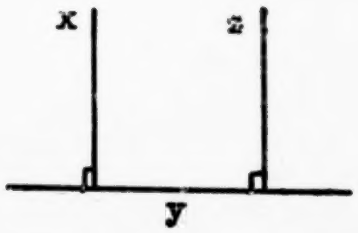
A (133, 158)

B (42 - 54, 75 - 81)

There are many theorems about the special relationships that exist between lines and planes. Time restrictions prohibit formal proof of all of them. However, it is important that pupils DEVELOP THE ABILITY TO VISUALIZE SPATIAL RELATIONSHIPS BETWEEN LINES AND PLANES.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
INDIRECT REASONING To develop an understanding of indirect reasoning as a method of proof	indirect reasoning — If all possible conclusions to a proposition are listed, and if all but one are proved false, then the remaining one is true. It is assumed that one of the listed conclusions must be true. Th. If two lines are perpendicular to a third line all in the same plane, the two lines are parallel.

REFERENCES	SUGGESTIONS
C (101 - 110, 163 - 172) E (168 - 171) F (153 - 157) G (13 - 21)	This can be done by using thin sticks and pieces of cardboard to represent lines and planes. Overlaying a transparency on the overhead projector is also helpful. Pupils can draw correct conclusions intuitively. Pupils' understanding of these spatial relationships can be tested by careful selection of exercises from any of the references available.
A (299, 305 - 307) C (113 - 114) D (181 - 182) E (138 - 139) F (163 - 166)	Present indirect reasoning as three steps. 1. List all possible conclusions, one of which must be true. 2. Prove that all conclusions except one lead to a contradiction of the hypothesis or contradict a statement previously proved true. 3. State that the one remaining conclusion must be true. Use indirect proof as a means of proving this theorem as an exercise.



Given: $x \perp y$, $z \perp y$,
all in the same plane.

To prove: $x \parallel z$.

Plan: Either $x \parallel z$ or $x \nparallel z$.
Assume $x \nparallel z$, and show that this leads to a contradiction. Then x must be parallel to z.

Proof

Statements	Reasons
1. $x \perp y$, $z \perp y$; x, y, z lie in the same plane.	1. Given
2. Either $x \parallel z$ or $x \nparallel z$.	2. Two straight lines in the same plane either intersect or do not intersect.
3. Assume $x \nparallel z$, then x and z will meet at some point P.	3. Two intersecting lines determine a point.
4. Then there are two perpendiculars from P to y.	4. Given $x \perp y$, and $z \perp y$.
5. This is impossible.	5. In a plane, one and only one line can be drawn through a point perpendicular to a given line.
6. $x \parallel z$	6. Since all other conclusions are false, the remaining conclusion must be true.

TOPICS AND OBJECTIVES

PARALLEL LINES

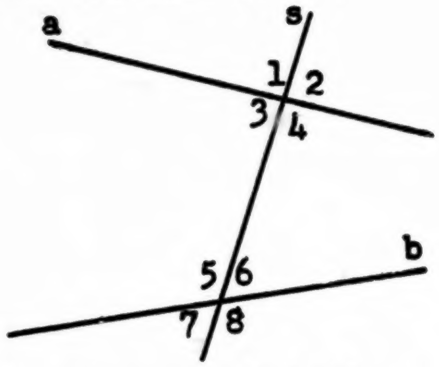
To develop comprehension of the vocabulary used in proving parallel line theorems

CONCEPTS, VOCABULARY, SYMBOLISM

Postulate: Through a given point not in a given line, one and only one line can be drawn parallel to the given line.

transversal A transversal is a line that intersects two or more straight lines. The word transversal will be used only if all the lines lie in the same plane.

s is the transversal of a and b.



Consider the eight numbered angles. Four of them, $\angle$s 3, 4, 5, and 6 are called interior angles.

Four of them, $\angle$s 1, 2, 7, and 8 are called exterior angles.

alternate interior angles (alt. int. $\angle$s) Alternate interior angles are pairs of non-adjacent interior angles that lie on opposite sides of the transversal.

alt. int. $\angle$s by pairs are: $\angle 3$ and $\angle 6$
$\angle 4$ and $\angle 5$

alternate exterior angles (alt. ext. $\angle$s) Alternate exterior angles are pairs of non-adjacent exterior angles that lie on opposite sides of the transversal.

alt. ext. $\angle$s by pairs are: $\angle 1$ and $\angle 8$
$\angle 2$ and $\angle 7$

corresponding angles (corr. $\angle$s) Corresponding angles are angles on the same side of the transversal and on the same side of the lines cut by the transversal. Note that one angle is an interior angle and the other is an exterior angle.

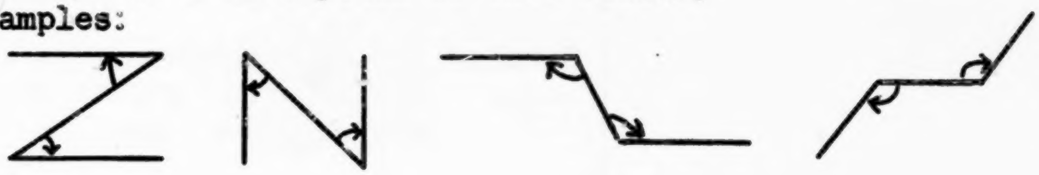
corr. $\angle$s by pairs are: $\angle 1$ and $\angle 5$
$\angle 3$ and $\angle 7$
$\angle 2$ and $\angle 6$
$\angle 4$ and $\angle 8$

interior angles on the same side of the transversal Interior angles on the same side of the transversal by pairs are:

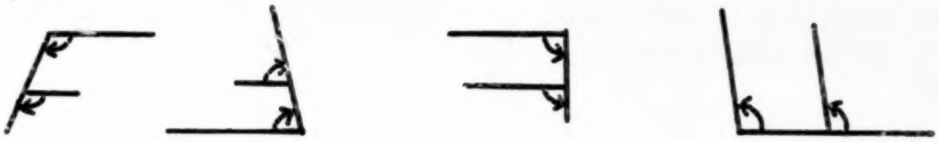
$\angle 4$ and $\angle 6$
$\angle 3$ and $\angle 5$

REFERENCES	SUGGESTIONS
A (141, 146) F (157 - 160)	Pupils can be aided in recognizing alternate interior angles, corresponding angles, and interior angles on the same side of the transversal in the more complex geometric figures by use of certain "code" letters.

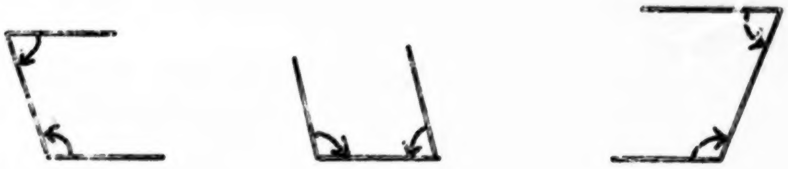
Alternate interior angles can be discovered as they form the letter "Z" or a corruption of this letter.
Examples:



Corresponding angles can be discovered as they form the letter "F" or a corruption of this letter.
Examples:



Interior angles on the same side of the transversal can be discovered as they form the letter "C" or a corruption of this letter.
Examples:



Have the pupils use colored pencils and mark the Z's, F's, and C's in their drawings.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PROVING LINES PARALLEL To develop the ability to prove lines parallel	**Th.** (prove formally) If two straight lines are cut by a transversal so that the alternate interior angles are equal, the lines are parallel. **Corol.** If two straight lines are cut by a transversal so that the corresponding angles are equal, the lines are parallel. **Corol.** If two straight lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, the lines are parallel. **Corol.** If two straight lines are cut by a transversal so that the alternate exterior angles are equal, the lines are parallel.
USING PARALLEL LINES TO PROVE ANGLE RELATIONSHIPS To develop an understanding of the theorems in which "given" parallel lines establish angle relationships	**Th.** If a line is perpendicular to one of two parallel lines, it is perpendicular to the other line. **Th.** If two parallel lines are cut by a transversal, the alternate interior angles are equal. **Corol.** If two parallel lines are cut by a transversal, the corresponding angles are equal. **Corol.** If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary. **Corol.** If two parallel lines are cut by a transversal, the alternate exterior angles are equal. **Corol.** If two lines are parallel to the same line, they are parallel to each other.

REFERENCES

SUGGESTIONS

A (143 - 144, 146 - 148)

C (119 - 121)

E (150 - 155)

F (167 - 173)

Have the pupils add to lists of methods of proof a summary of the ways of proving lines parallel.

A (149 - 155, 159 - 160)

C (122 - 126)

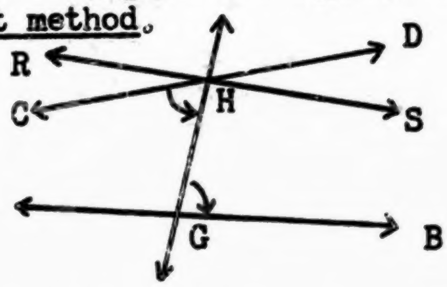
D (82 - 90)

E (144 - 147)

F (158 - 163)

Have pupils discover the angle relationships when given parallel lines. Use the ruled lines on a sheet of note paper as parallel lines. Draw a transversal and have the pupils measure the angles with a protractor.

The theorem, "If two parallel lines are cut by a transversal, the alternate interior angles are equal," may be proved through use of the indirect method.



Summarizing:

The alternate interior angles are equal or not equal.
If the angles are assumed not equal, then there must be a line RHS, distinct from CD, such that $\angle RHG = \angle HGB$.
Then line RHS is parallel to line AB, since the alternate interior angles are equal.
This means that there are two lines through point H parallel to AB.
But this contradicts the parallel line postulate, "Through a given point not in a given line, one and only one line can be drawn parallel to the given line."
Therefore, the given alternate interior angles must be equal.

Have pupils add these theorems and corollaries concerning parallel lines to the list of methods of proving angles equal.

UNIT V

POLYGONS AND POLYHEDRONS

20 Sessions

Unit V - Polygons and Polyhedrons (20 sessions)

TOPICS AND OBJECTIVES

CONCEPTS, VOCABULARY, SYMBOLISM

PROPERTIES OF POLYGONS

To develop an understanding of the vocabulary pertaining to polygons

broken line

A broken line is the union of successive line segments such that:

1. the successive segments are not in a straight line.
2. no more than two line segments have a common end point.

adjacent segments

Adjacent segments are two successive line segments with a common end point.

closed broken line

A broken line is closed if each line segment is adjacent to a successive line segment at each of its end points.

polygon

A polygon is a closed broken line in a plane.
The line segments are the sides of the polygon.
The end points of the line segments are the vertices of the polygon.
A diagonal of a polygon is a line segment joining nonadjacent vertices.

A closed broken line separates the plane into three disjoint subsets.

Set A = all the points in the polygon
Set B = all the points in the interior of the polygon
Set C = all the points not in the polygon nor in the interior of the polygon
$A \cup B$ = the polygonal region

convex polygon

A convex polygon is a polygon no side of which extended will enter the interior of the polygon.
Each of the interior angles is less than a straight angle.

concave polygon

A concave polygon is a polygon having at least one side which extended will enter the interior of the polygon.
One or more of the interior angles is a reflex angle.

Unless otherwise indicated, all polygons are to be considered as convex polygons.

REFERENCES	SUGGESTIONS
A (161) C (129 - 130) D (98 - 100) E (12, 215 - 216)	

CLASSIFICATION OF POLYGONS ACCORDING TO SIDES

To develop the ability to classify polygons according to the number of sides

triangle	a polygon with three sides
quadrilateral	a polygon with four sides
pentagon	a polygon with five sides
hexagon	a polygon with six sides
heptagon	a polygon with seven sides
octagon	a polygon with eight sides
nonagon	a polygon with nine sides
decagon	a polygon with ten sides
dodecagon	a polygon with twelve sides
pentadecagon	a polygon with fifteen sides
heptadecagon	a polygon with seventeen sides
n-gon	a polygon with "n" sides

equilateral polygon An equilateral polygon is a polygon all of whose sides are equal.

equiangular polygon An equiangular polygon is a polygon all of whose angles are equal.

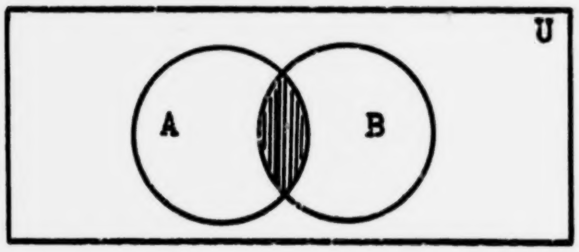
regular polygon A regular polygon is a polygon all of whose sides are equal and all of whose angles are equal.
The set of regular polygons is the intersection of the sets of equilateral and equiangular polygons.

THEOREMS INVOLVING POLYGONS

To develop the ability to prove certain theorems involving polygons

Th. (prove formally) The sum of the angles of a triangle is equal to a straight angle (180°).

Corol. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.



U = the set of all polygons
A = the set of all the equilateral polygons
B = the set of all the equiangular polygons
$A \cap B$ = the set of all regular polygons

A (162 - 168)

C (131 - 139)

D (109 - 110)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Corol. A triangle can have at most one right angle or one obtuse angle.
	Corol. The acute angles of a right triangle are complementary.
	Corol. Each angle of an equilateral triangle contains 60°.
	Corol. If two angles and a side of one triangle are equal respectively to two angles and a side of another triangle, the triangles are congruent. (a.a.s. = a.a.s.)
	Corol. If one side of a triangle is extended, the exterior angle thus formed is equal to the sum of the two remote interior angles.
	exterior angle of a triangle: An exterior angle of a triangle is the angle formed by one side of the triangle extended through a vertex, and an adjacent side.

REFERENCES

E (156 - 161)

F (176 - 180)

A (156 - 157)

C (116)

D (111)

E (142)

F (167, 187)

Bell, E. T. Men of Mathematics. (218-269, 294-306, 484-509)

Bergamini, David and the Editors of Life. Mathematics. (155-167)

Dresden, Arnold. An Invitation to Mathematics.

Eves, Howard. An Introduction to the History of Mathematics. (132-134)

Kasner, Edward and Newman, James. Mathematics and the Imagination. (131-150)

Kline, Morris. Mathematics in Western Culture.

SUGGESTIONS

NON-EUCLIDEAN GEOMETRY

The acceptance of the proof of the theorem, "The sum of the angles in a triangle equals a straight angle," as well as many other theorems involving the use of parallel lines in their proof, is dependent upon Euclid's fifth postulate.
This is the famous "parallel postulate".
Upon examination of Euclid's original postulates, one has the feeling that the fifth is not as "self-evident" as the others.

Euclid himself was unable to prove it as a theorem. While he was also unable to disprove it, there is evidence to the fact that in later years he was dissatisfied with this postulate and wanted to divorce it from the others.
Mathematicians throughout history have attempted to prove or disprove this postulate, all unsuccessfully.

Three mathematicians--Lobatchevsky of Russia, Bolyai of Hungary, and Gauss of Germany--working independently, replaced the "parallel postulate" with one that states, "Through any point not in a given line more than one line can be drawn parallel to the given line."

From this postulate developed a geometry which is every bit as valid and consistent as that of Euclid.
In fact, these geometries are identical with Euclidean geometry excepting those theorems dependent upon the parallel postulate.
For example, one startling difference is: "The sum of the angles in a triangle is less than a straight angle."
Gauss did much of the early work in this field but failed to communicate his findings to the world. As a result, most of the credit is given to Bolyai and particularly to Lobatchevsky.
This branch of non-Euclidean geometry is known as Lobatchevskian or "hyperbolic" geometry since the nature of the geometry is best suited to a hyperbolic surface (a surface with constant negative curvature) rather than to a plane surface.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. The sum of the interior angles of a polygon with n sides is equal to 180°(n - 2).
	Corol. The sum of the exterior angles of a polygon made by extending each of its sides in succession is 360°.

REFERENCES

Lieber, Lillian. Non-Euclidean Geometry or Three Moons in Mathesis.

Smith, D. E. History of Mathematics, Vol. II. (335 – 338)

A (170 – 173)

C (140 – 143)

D (112 – 114)

E (218 – 221)

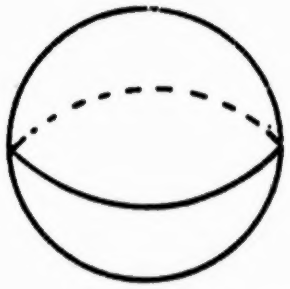
F (63 – 64)

SUGGESTIONS

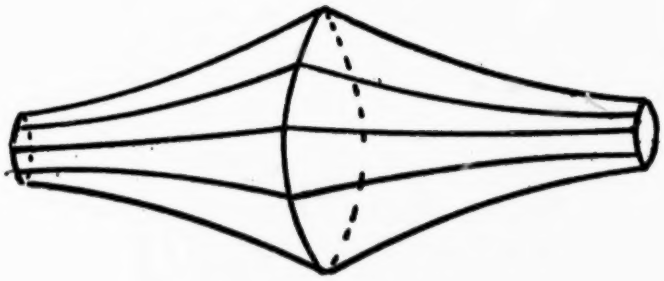
A little later, Riemann, another German mathematician, replaced the parallel postulate with one that states, "Through a point not in a line, no line can be drawn parallel to the given line." This postulate leads to the theorem, "The sum of the angles of a triangle is greater than a straight angle," as well as other contradictory theorems. Here again, the geometry is consistent and valid and has turned out to be more practical than Euclid in many modern applications.
Einstein used Riemannian geometry in his theory of relativity. It is also known as "elliptic" geometry (applied to a surface of constant positive curvature) and is very similar to geometry applied to the surface of a sphere.

Non-Euclidean geometry, in addition to being a worth-while logical exercise for pupils in abstract geometry, plays a vital role in advanced mathematics and science. The question, "Is space curved?" may well be restated, "Is space best described by Euclidean, Lobatchevskian, or Riemannian geometry?"

See appendix for a table comparing features of Euclidean and non-Euclidean geometries.



Sphere – A surface on which, with restrictions, Riemannian geometry may be pictured.



Pseudosphere – A surface on which Lobatchevskian geometry may be pictured. A simple hyperboloid may also be used as a surface.

QUADRILATERALS

To develop an understanding of the definitions for different types of quadrilaterals

trapezium A trapezium is a quadrilateral with no sides parallel. A trapezium is the general form for a quadrilateral.

trapezoid A trapezoid is a quadrilateral with two and only two sides parallel.
If the non-parallel sides are equal, the trapezoid is called an isosceles trapezoid.
If a trapezoid has one right angle, it is called a right trapezoid.
The median of a trapezoid is a line segment joining the midpoints of the non-parallel sides.
The bases of a trapezoid are the parallel sides.

parallelogram A parallelogram is a quadrilateral with opposite sides parallel.
The symbol for parallelogram is ▱.
Properties of a parallelogram:

1. opposite sides are equal
2. opposite angles are equal
3. successive angles are supplementary
4. diagonals bisect each other
5. either diagonal divides the parallelogram into two congruent triangles

METHODS OF PROVING THAT A QUADRILATERAL IS A PARALLELOGRAM

To develop an awareness of the theorems which prove that a quadrilateral is a parallelogram

Th. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

Th. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

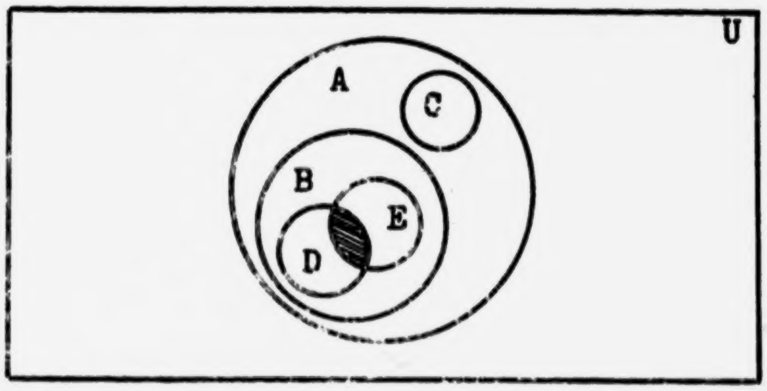
Th. If two sides of a quadrilateral are parallel and equal, the figure is a parallelogram.

Th. If the successive angles of a quadrilateral are supplementary, the figure is a parallelogram.

Th. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

REFERENCES	SUGGESTIONS
A (174 - 180) **C** (151 - 154, 155 - 161) **D** (151 - 159) **E** (222 - 228) **F** (65 - 67)	**Sidelight.** In Great Britain, the definitions of trapezium and trapezoid are interchanged. It is not necessary to prove any theorems or exercises concerning the properties of parallelograms, methods of proving that quadrilaterals are parallelograms, or the special properties of rectangles, rhombuses and squares. It is better to have the pupils "research" these items, compile lists of their own, and then formulate correct lists cooperatively. When complete and correct lists have been compiled, pupils may use these properties and methods as acceptable reasons for proofs.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
SPECIAL TYPES OF PARALLELOGRAMS	
To develop the ability to recognize special types of parallelograms and to define the properties of each	**rectangle** A rectangle is a parallelogram with one right angle. A rectangle has all the properties of a parallelogra plus the following special properties: 1. has four right angles 2. diagonals are equal. **rhombus** A rhombus is a parallelogram with two adjacent sides equal. A rhombus has all the properties of a parallelogram plus the following special properties. 1. all sides are equal 2. the diagonals are perpendicular to each other 3. the diagonals bisect the angles 4. both diagonals divide the rhombus into four congruent triangles. **square** A square is a parallelogram with one right angle and with two adjacent sides equal. A square has all the properties of a parallelogram plus: 1. all the special properties of a rectangle 2. all the special properties of a rhombus.
SPECIAL THEOREMS PERTAINING TO CERTAIN POLYGONS	
To develop an understanding of theorems concerning polygons	Th. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

REFERENCES	SUGGESTIONS
	Below is a Venn diagram of the family of polygons with special reference to quadrilaterals.  U = the set of all polygons A = the set of all quadrilaterals B = the set of all parallelograms C = the set of all trapezoids D = the set of all rectangles E = the set of all rhombuses D ∩ E = the set of all squares
A (184 - 191, 193 - 194, 201 - 205)	This theorem may be proved as an exercise.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. If a line segment joins the midpoints of two sides of a triangle, it is parallel to the third side and equal to half of it. Corol. If a line bisects one side of a triangle and is parallel to the second side, it bisects the third side.

REFERENCES	SUGGESTIONS
C (154 - 155, 159 - 161, 182 - 185) D (159 - 165) E (230 - 236)	Add this theorem to the list of methods for proving lines parallel.

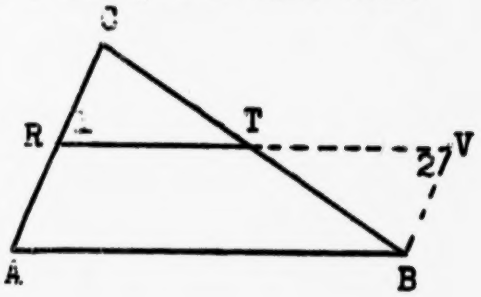
An alternate method for proving theorems is to begin at the conclusion and think backward until reaching the hypothesis. This method of discovering a proof is called the analytic method of attack.

Example: If a line joins the midpoints of two sides of a triangle:
a. it is parallel to the third side
b. it is equal to half of the third side.

Given: $\triangle$ ABC with R and T, the midpoints of AC and BC, respectively.

To prove: a. $RT \parallel AB$
b. $RT = \frac{1}{2}(AB)$

Plan: Extend RT its own length to V and prove that RVBA is a parallelogram.



Begin at conclusion (b) and ask,
"How can $RT = \frac{1}{2}(AB)$?"
Since $RT = \frac{1}{2}(RV)$ by construction,
$RT = \frac{1}{2}(AB)$ if $AB = RV$.
Begin at conclusion (a) and ask,
"How can it be proved that $RT \parallel AB$?"
Since RT is part of RV, $RT \parallel AB$ if $RV \parallel AB$.
$RV \parallel AB$ and $RV = AB$ if ABVR is a parallelogram.
ABVR is a parallelogram if $AR = BV$ and $AR \parallel BV$.
Since AR is part of AC, $AR \parallel BV$ if $AC \parallel BV$.
$AC \parallel BV$ if $\angle 1 = \angle 2$.
$\angle 1 = \angle 2$ if $\triangle RCT \cong \triangle VBT$.
$\triangle RCT \cong \triangle VBT$ if s.a.s. = s.a.s.
This is true since $\angle CTR = \angle VTB$, $CT = TB$, and $RT = TV$.
Since $AR = RC$, $AR = BV$ if $RC = BV$.
$RC = BV$ if $\triangle RCT \cong \triangle VBT$.
This has already been proved so $RC = BV$ since C.p.c.t.e.
Reversing these steps will prove the exercise.

Pupils having difficulty with the proof of a problem will often be able to clear up any difficulty by the analytic method of attack.

Th. The median of a trapezoid is parallel to the bases and equal to one-half their sum.

Th. The base angles of an isosceles trapezoid are equal.

Corol. The diagonals of an isosceles trapezoid are equal.

Th. If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on every transversal.

POLYHEDRONS

To develop an understanding of basic concepts regarding geometric solids

polyhedron — A polyhedron is a solid formed by a set of planes (four or more) which enclose a region of space.
The planes are called the faces of the polyhedron.
These faces are enclosed by polygons.
Thus, the bases of the polygon are polygonal regions.
The intersection of the faces are the edges of the polyhedron.
The edge of a polyhedron is the edge of the dihedral angle formed by the intersection of any two faces.
The vertices of a polyhedron are the points where three or more edges intersect.
A diagonal of a polyhedron is a line segment joining any two vertices not in the same face.

The polyhedron separates space into three disjoint subsets.

Set A = all the points in the polyhedron
Set B = all the points in the interior of the polyhedron
Set C = all the points not in the polyhedron nor in the interior of the polyhedron
$A \cup B$ = the polyhedral region

convex polyhedron — A polyhedron is convex if every edge extended does not enter the interior region of the polyhedron.
Unless otherwise indicated, every polyhedron will be considered as being convex.

section of a solid — A plane figure which is formed by the intersection of a plane and a solid is called a section of the solid.
The section of a polyhedron is a polygon.

REFERENCES	SUGGESTIONS
	Add this theorem to the list of methods for proving lines parallel. An excellent fallacy problem which should be used for enrichment purposes can be found in the appendix. The fallacy seems to prove, "A right angle is equal to an angle greater than a right angle!" The fallacy in the problem becomes readily apparent when the figure is accurately drawn.
A (200) B (61 - 62) C (168) D (255 - 256) E (126, 240) G (27, 29)	The word "polyhedron" means "many planes".

polyhedral angle A polyhedral angle is a figure formed by three or more planes that meet in a point. The planes must be so situated that they may be intersected by another plane, the section formed being a polygon.

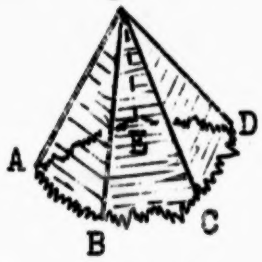
The meeting point of the planes is called the vertex of the polyhedral angle.
The portions of the planes which form the polyhedral angle are called the faces.
A face angle of the polyhedral angle is a plane angle formed by the edges of any one face.

A polyhedral angle is named by naming the vertex point alone, or the vertex point and a point on each edge.

Polyhedral ∠T

or

Polyhedral ∠T-ABCDE



∠ATB and ∠BTC are examples of face angles.

The measure of a polyhedral angle is equal to the sum of the measures of its face angles.

A polyhedral angle having three faces is called a trihedral angle.
Polyhedral angles of four, five, six, and eight faces respectively are called tetrahedral, pentahedral, hexahedral, and octahedral angles.

regular polyhedron A regular polyhedron is a polyhedron all of whose faces are congruent regular polygons and all of whose polyhedral angles are equal.

There are only five regular polyhedrons:
1. regular tetrahedron - four equilateral triangles
2. regular hexahedron or cube - six squares
3. regular octahedron - eight equilateral triangles
4. regular dodecahedron - twelve regular pentagons
5. regular icosahedron - twenty equilateral triangles

REFERENCES	SUGGESTIONS
	Other definitions for polyhedral angle are: 1. A figure generated by the rotation of a ray about its end point while intersecting a polygon in another plane. 2. The configuration formed by the lateral faces of a polyhedron which have a common vertex. 3. The figure formed by the union of a point and the rays joining that point to each point of the sides of a polygon in a plane not containing the point.
D (257 - 260)	The five regular polyhedrons are also known as the Platonic Solids in honor of their discoverer, Plato.
Cundy, H. M. and Rollett, A. P. Mathematical Models. (77-160) Gamow, G. One, Two, Three--Infinity!	There are many polyhedrons whose faces are regular polygons and whose polyhedral angles are equal but whose faces are not all the same kind of regular polygon. These are known as Archimedian polyhedrons. An example is the great rhombicosidodecahedron. This substantial solid consists of 62 faces, 30 of which are squares, 20 of which are regular hexagons, and 12 of which are regular decagons.

prismatoid — A prismatoid is a polyhedron all of whose vertices lie in two parallel planes.

PRISMS

To develop the ability to classify prisms as polyhedrons

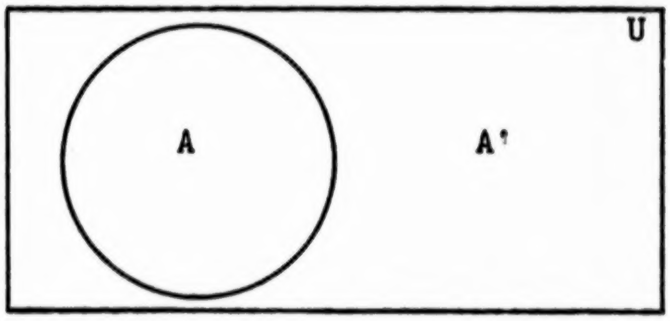
prism — A prism is a polyhedron in which two faces, called bases, are congruent polygons which lie in parallel planes.
The other faces of the prism are parallelograms and are called lateral faces.
The intersection of any two lateral faces is called a lateral edge.

Prisms can be classified according to their bases.

- A prism whose bases are triangles is called a triangular prism.
- A prism whose bases are quadrilaterals is called a quadrangular prism.
- A prism whose bases are hexagons is called a hexagonal prism.
- A prism whose bases are octagons is called an octagonal prism.

right prism — A right prism is a prism whose lateral edges are perpendicular to the bases.

oblique prism — An oblique prism is a prism whose lateral edges are not perpendicular to the bases.

REFERENCES	SUGGESTIONS
Hogben, L. Mathematics in the Making. (286-287, 291-294)	Have pupils discuss the regular polyhedrons, determining such features as the number of vertices and edges, the number of degrees in each polyhedral angle, and so on.
Young, F. H. The Nature of Regular Polyhedra · Infinity and Beyond · An Introduction to Groups. (1-8)	In the chapter on inequalities, pupils will be able to prove that there are only five regular polyhedrons. Introduce Euler's Theorem: In any polyhedron which has no holes, the sum of the number of faces and the number of vertices is equal to two more than the number of edges. The formula is $V + F = E + 2$. Pupils may check this formula first using the regular polyhedrons and then any irregular polyhedrons. The proof of Euler's Theorem makes a good project. Film: Stretching the Imagination (30 min.) Association Films, Inc. 347 Madison Avenue New York 17, New York One of the series, "Adventures in Number and Space". Bill Baird and his puppets discuss topology and Euler's theorem.
A (200) B (84 - 91) D (261 - 268) E (240 - 243) G (30 - 33)	The relationship among the various polyhedrons may be illustrated by Venn diagrams.  U = the set of all prisms A = the set of all right prisms A' = the set of all oblique prisms (Set A' is the complement of set A)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

regular prism	A regular prism is a right prism whose bases are regular polygons.
parallelepiped	A parallelepiped is a prism whose bases are parallelograms.
right parallelepiped	A right parallelepiped is a parallelepiped which is a right prism.
rectangular parallelepiped	(rectangular prism, rectangular solid) A rectangular parallelepiped is a right parallelepiped whose bases are rectangles.
cube	A cube is a rectangular parallelepiped whose faces are squares.
right section	A right section of a prism is the figure formed by the intersection of a plane perpendicular to the lateral edges.

PROPERTIES OF PRISMS

To develop an understanding of the properties of prisms

Postulate: Sections of a prism made by parallel planes which cut off the lateral edges or these edges extended are congruent polygons.

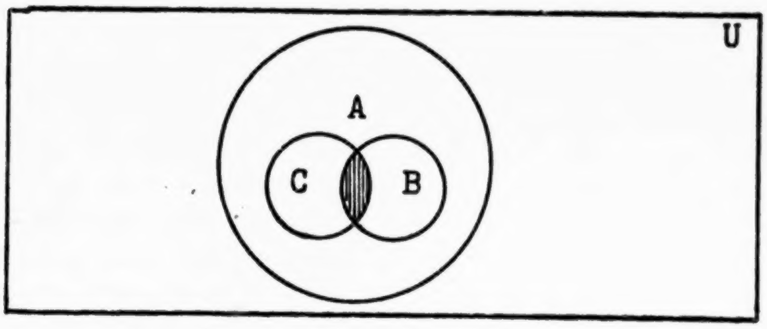
Th. Every section of a prism made by a plane parallel to a base is congruent to that base.

Th. The opposite faces of a parallelepiped are parallel and congruent polygons.

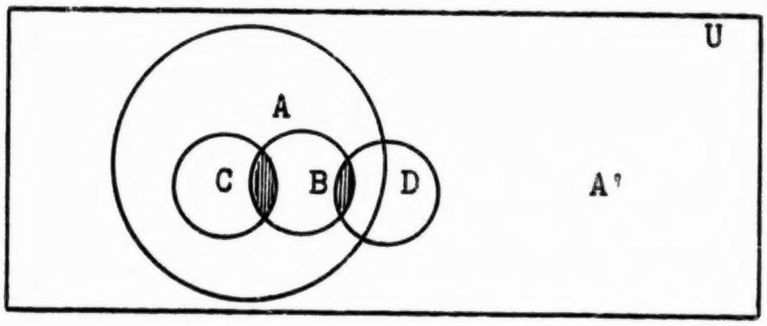
Th. The lateral faces of a right prism are rectangles.

Th. The altitude of a right prism equals the lateral edge.

REFERENCES	SUGGESTIONS



U = the set of all prisms
A = the set of all right prisms
B = the set of all regular prisms
C = the set of all rectangular parallelepipeds
$B \cap C$ = the set of all cubes



U = the set of all prisms
A = the set of all right prisms
A'= the set of all oblique prisms
B = the set of all regular prisms
C = the set of all rectangular parallelepipeds
$B \cap C$ = the set of all cubes
D = the set of all triangular prisms
$B \cap D$ = the set of all regular triangular prisms

It is not necessary to prove any of the theorems in connection with the properties of prisms and pyramids. It is quite sufficient to postulate these properties.
Pupils will be able to grasp intuitively the necessary spatial concepts.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PYRAMIDS	
To develop the ability to classify pyramids as polyhedrons	**pyramid** A pyramid is a polyhedron with one face a polygon and the other faces triangles with a common vertex. The polygon is called the base of the pyramid and the triangles are called the lateral faces. The common vertex of the lateral faces is called the vertex of the pyramid. The intersections of pairs of lateral faces are the lateral edges. The perpendicular distance from the vertex to the base is called the altitude. A pyramid is classified by the type of polygon which is its base. A pyramid whose base is a triangle is a triangular pyramid. A pyramid whose base is a square is called a square pyramid. **regular pyramid** A regular pyramid is a pyramid whose base is a regular polygon and whose lateral faces are congruent triangles. The slant height of a regular pyramid is the altitude of any of its lateral faces.
PROPERTIES OF PYRAMIDS	
To develop an understanding of the properties of pyramids	Th. The slant heights of a regular pyramid are equal. Th. The lateral faces of a regular pyramid are enclosed by congruent isosceles triangles. Th. The altitude of a regular pyramid passes through the center of the base. **frustum of a pyramid** A frustum of a pyramid is the portion of a pyramid between the base and a plane parallel to the base. The bases of the frustum are: 1. the base of the pyramid 2. the section made by the intersection of the plane and the pyramid. The lateral faces of a frustum of a pyramid are trapezoids. **truncated pyramid** A truncated pyramid is the portion of a pyramid between the base and a plane oblique to the base. The bases are: 1. the base of the pyramid 2. the section made by the intersection of the plane and the pyramid. The lateral faces of a truncated pyramid are trapeziums.

REFERENCES		SUGGESTIONS
A	(200)	The teacher should use any models available. The teacher will need to use good judgment in selecting an adequate number of appropriate exercises from the references listed.
B	(111 - 115)	
C	(10, 277)	
D	(270 - 275)	
E	(276 - 277, 444 - 445)	
F	(276, 518)	
G	(40 - 42)	

UNIT VI

INEQUALITIES

10 Sessions

Unit VI - Inequalities (10 sessions)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
INEQUALITIES To develop an understanding of the terminology for the order of inequalities	inequalities of the same order — Two inequalities are of the same order if the same inequality sign is used in both inequalities. $6 > 5$ is in the same order as $7 > 2$. inequalities of the opposite order — Two inequalities are of the opposite order if the greater inequality sign is in one inequality and the lesser inequality symbol is in the other inequality. $6 > 5$ is in opposite order from $2 < 7$.
AXIOMS OF INEQUALITY To develop an understanding of the axioms of inequality	1. Addition axiom (equal and unequal quantities) If equal quantities are added to unequal quantities, the sums are unequal in the same order. 2. Addition axiom (unequal quantities) If unequal quantities are added to unequal quantities of the same order, the sums are unequal in the same order. 3. Subtraction axiom (unequal quantities minus equal quantities) If equal quantities are subtracted from unequal quantities, the remainders are unequal in the same order. 4. Subtraction axiom (equal quantities minus unequal quantities) If unequal quantities are subtracted from equal quantities, the remainders are unequal in the opposite order. 4a. Corol. Supplements or complements of unequal angles are unequal in the opposite order.

REFERENCES	
A	(300 - 303)
C	(197 - 199)
D	(174 - 178)
E	(487 - 490)
F	(101)

SUGGESTIONS

Examples:

1. If $a > b$, then $a + x > b + x$.

$$\begin{array}{r} 8 > 5 \\ + \underline{(2 = 2)} \\ 10 > 7 \end{array}$$

2. If $a > b$, and $c > d$, then $a + c > b + d$.

$$\begin{array}{r} 5 > 2 \\ + \underline{(9 > 8)} \\ 14 > 10 \end{array}$$

3. If $a > b$, then $a - x > b - x$.

$$\begin{array}{r} 9 > 6 \\ - \underline{(4 = 4)} \\ 5 > 2 \end{array}$$

4. If $a > b$, then $x - a < x - b$.

$$\begin{array}{r} 7 = 7 \\ - \underline{(5 > 3)} \\ 2 < 4 \end{array}$$

4a. If $\angle A > \angle B$, then $(90^\circ - \angle A) < (90^\circ - \angle B)$.
If $\angle A < \angle B$, then $(180^\circ - \angle A) > (180^\circ - \angle B)$.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	5. **Multiplication axiom** If unequal quantities are multiplied by positive equal quantities, the products are unequal in the same order. If unequal quantities are multiplied by negative equal quantities, the products are unequal in the opposite order.
	6. **Division axiom** If unequal quantities are divided by positive equal quantities, the quotients are unequal in the same order. If unequal quantities are divided by negative equal quantities, the quotients are unequal in the opposite order.
	7. **Transitive axiom** If three quantities are so related that the first is greater than the second and the second is greater than the third, then the first is greater than the third.
	8. **Powers and roots axiom** Equal positive powers of positive unequal quantities are unequal in the same order. Equal positive roots of positive unequal quantities are unequal in the same order.
	9. **Axiom of the whole** The whole is greater than any of its parts.

REFERENCES	SUGGESTIONS

5. If $a > b$, then $ax > bx$, where $x > 0$.

$$\begin{array}{r} 4 > 3 \\ \times\ \underline{(5 = 5)} \\ 20 > 15 \end{array}$$

If $a > b$, then $ax < bx$, where $x < 0$.

$$\begin{array}{r} 4 > 3 \\ \times\ \underline{(-5 = -5)} \\ -20 < -15 \end{array}$$

6. If $a > b$, then $\frac{a}{x} > \frac{b}{x}$, where $x > 0$. $\frac{14}{2} > \frac{8}{2}$

If $a > b$, then $\frac{a}{x} < \frac{b}{x}$, where $x < 0$. $\frac{14}{-2} < \frac{8}{-2}$

7. If $a > b$ and $b > c$, then $a > c$. $8 > 5$ and $5 > 3$, then $8 > 3$.

8. If $a > b$ and $a > 0$ and $b > 0$, then $a^2 > b^2$.
$25 > 16$, then $625 > 256$.

If $a > b$ and $a > 0$ and $b > 0$, then $\sqrt{a} > \sqrt{b}$.
$25 > 16$, then $5 > 4$.

9. If $a = b + k$, then $a > b$ and $a > k$.
$9 = 4 + 5$, then $9 > 4$ and $9 > 5$.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
THEOREMS OF INEQUALITY	
To develop an understanding of theorems involving inequalities	Th. (prove formally) If one side of a triangle is extended, the exterior angle formed is greater than either of the remote interior angles.

REFERENCES

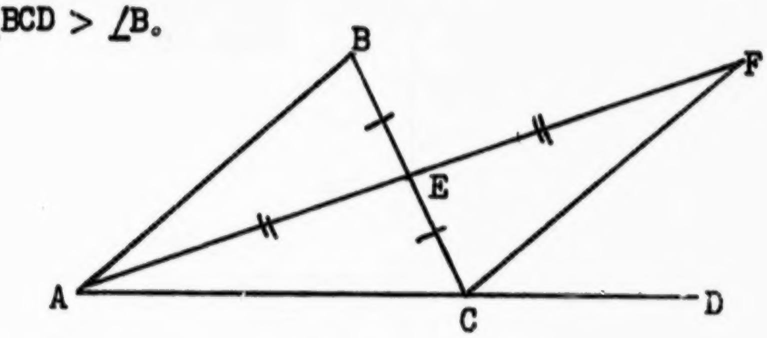
A	(303 - 304, 308 - 311)
C	(200 - 203)
D	(178 - 183)
E	(490 - 495)

SUGGESTIONS

Th. If one side of a triangle is extended, the exterior angle formed is greater than either of the remote interior angles.

Given: △ ABC with AC extended to D.

To prove: $\angle BCD > \angle B$.



Plan: Let E be the midpoint of BC.
Draw AE and extend AE to F so that AE = EF.
Draw CF.

Proof

	Statements		Reasons
1.	BE = EC	1.	A midpoint divides a line segment in two equal parts.
2.	AE = EF	2.	So drawn
3.	$\angle BEA = \angle CEF$	3.	Vertical angles are equal.
4.	△ AEB ≅ △ FEC	4.	s.a.s. = s.a.s.
5.	$\angle B = \angle ECF$	5.	C.p.c.t.e.
6.	$\angle BCD = \angle ECF + \angle FCD$	6.	Axiom of the whole (equalities)
7.	$\angle BCD = \angle B + \angle FCD$	7.	Substitution axiom
8.	$\therefore \angle BCD > \angle B$	8.	Axiom of the whole (inequalities)

The proof that $\angle BCD > \angle BAC$ is similar to the above proof. This theorem is proved without the use of the corollary, "The exterior angle of a triangle is equal to the sum of the remote interior angles," and should be presented to the pupils in the above manner.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. If two sides of a triangle are unequal, the angles opposite those sides are unequal in the same order. Th. If two angles of a triangle are unequal, the sides opposite those angles are unequal in the same order. Corol. The sum of any two sides of a triangle is greater than the third side. Th. The sum of the face angles of a polyhedral angle is less than 360°.
	Th. The sum of any two face angles in a trihedral angle is greater than the third face angle. Corol. The sum of any (n - 1) face angles in a polyhedral angle with n faces is greater than the nth face angle.

These theorems may be proved indirectly as an exercise.

(324)

(62 - 65)

(88 - 91)

To help pupils see this theorem intuitively, a simple but very helpful model may be made.
Use a piece of plywood board of approximately equal length and width, rubber bands or elasticized string, and thumbtacks.

- Fasten three thumbtacks to the board to represent three non-collinear points A, B, and C in plane M. (See Figure #1.)
- Use three broken rubber bands or three pieces of elasticized string.
- Tie one end of each piece to each of the three tacks.
- Tie the other ends together at point D and make a loop at this conjuncture.
- Grasp the loop between the thumb and forefinger and pull it up and down.

  This will show that while the three lines meeting at D are in the plane, the sum of the angles formed is 360°. But as soon as a polyhedral angle is formed by pulling the loop up, the sum will be less than 360°. (See Figure #2.)

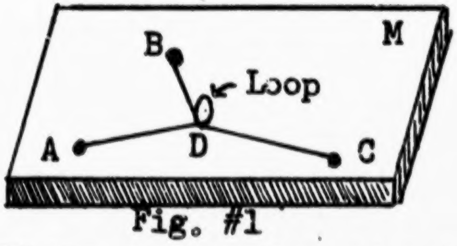


Fig. #1

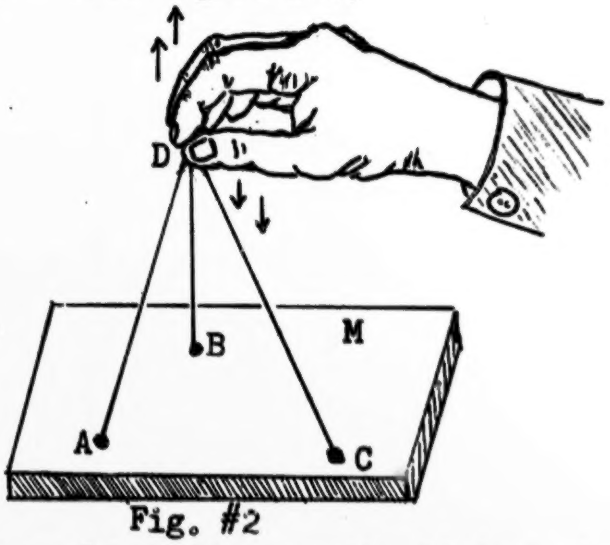


Fig. #2

If two face angles of a trihedral angle are known, the limits of the third face angle can be found as follows:

1. Lower limit: Subtract the smaller of the two known face angles from the larger.
2. Upper limit: Add the two face angles.
   a. If the sum of the known face angles is less than 180°, the sum is the upper limit.
   b. If the sum of the known face angles is greater than 180°, subtract this sum from 360° to obtain the upper limit.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

REFERENCES	SUGGESTIONS

Young, F. H. The Nature of Regular Polyhedra · Infinity and Beyond · An Introduction to Groups. (1-8)

The concept of a limit may be explained concisely by stating that a quantity may come very close to being a certain amount but can never quite reach that amount.

In testing whether any combination of angles can be face angles of a polyhedral angle, ask:

1. Is the sum of all the angles < 360°?
2. Is the sum of any (n - 1) face angles > the nth face angle?

Have pupils complete the second and third columns of this table.

Can a polyhedral angle be formed from:	Sum of the face angles	Answer
three equilateral triangles?	180°	Yes
four equilateral triangles?	240°	Yes
five equilateral triangles?	300°	Yes
six equilateral triangles?	360°	No
seven equilateral triangles?	420°	No
three squares?	270°	Yes
four squares?	360°	No
five squares?	450°	No
three regular pentagons?	324°	Yes
four regular pentagons?	432°	No
three regular hexagons?	360°	No
four regular hexagons?	480°	No
three regular heptagons?	$385\frac{5}{7}°$	No

From this table, pupils will conclude that it is impossible to form a polyhedral angle using more than five equilateral triangles, more than three squares, or more than three regular pentagons. No polyhedral angles may be formed using all regular hexagons, heptagons, octagons, and so on.
This development should lead pupils to conclude that there are only five regular polyhedrons.

UNIT VII

RATIO AND PROPORTION

5 Sessions

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
RATIO AND PROPORTION	
To reinforce the basic concepts of ratio and proportion	**ratio** The ratio of one quantity to another quantity is their quotient. The quotient is obtained by dividing the first quantity by the second quantity. One does *not* find the ratio of one *object* to another but rather the ratio of two *numbers* which are the measures of the objects. The symbol for ratio is ":". The ratio of a to b may be written $\frac{a}{b}$ or a:b.
	proportion A proportion is a statement of equality of two ratios. Four quantities are in proportion when the ratio of the first pair equals the ratio of the second pair. This is written as: $\frac{a}{b} = \frac{c}{d}$ or $a:b = c:d$. It is read as: a divided by b equals c divided by d or a is to b as c is to d. In this proportion a, b, c, and d are respectively the first, second, third, and fourth terms.
	extremes The first and fourth terms are called the extremes of the proportion.
	means The second and third terms are called the means of the proportion.

REFERENCES	SUGGESTIONS
A (327 - 329, 334, 336 - 341)	
C (205 - 207)	
D (309 - 316)	
E (259 - 260, 263 - 267)	
F (229 - 237)	

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PROPERTIES OF PROPORTIONS	
To develop an appreciation of the relationships concerning proportions	1. In any proportion, the product of the extremes is equal to the product of the means. 2. If the product of two non-zero numbers is equal to the product of two other non-zero numbers, the members of one pair may be made the means in a proportion and the members of the other pair may be made the extremes of a proportion. 3. If the numerators of a proportion are equal, the denominators are equal. The converse of this is true. 4. If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, the remaining terms are equal. 5. The terms of a proportion are in proportion by alternation; that is, the first term is to the third as the second is to the fourth. 6. The terms of a proportion are in proportion by inversion; that is, the second term is to the first as the fourth is to the third. 7. The terms of a proportion are in proportion by addition; that is, the sum of the first and second terms is to the second term as the sum of the third and fourth terms is to the fourth. 8. The terms of a proportion are in proportion by subtraction; that is, the first term minus the second is to the second as the third term minus the fourth is to the fourth. 9. In a series of equal ratios, the ratio of the sum of the numerators to the sum of the denominators is equal to the ratio of any numerator to its denominator.

REFERENCES

SUGGESTIONS

1. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

2. If $ef = gh$, then $\frac{e}{g} = \frac{h}{f}$ or $\frac{g}{e} = \frac{f}{h}$, as well as others.

3. If $\frac{k}{x} = \frac{k}{y}$, then $x = y$. If $\frac{x}{k} = \frac{y}{k}$, then $x = y$.

4. If $\frac{a}{b} = \frac{x}{c}$, and $\frac{a}{b} = \frac{y}{c}$, then $x = y$.

5. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

6. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

7. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a + b}{b} = \frac{c + d}{d}$.

8. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a - b}{b} = \frac{c - d}{d}$.

9. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots = \frac{m}{n}$, then
$\frac{a + c + e + \ldots m}{b + d + f + \ldots n} = \frac{a}{b} = \ldots = \frac{m}{n}$.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM	
	fourth proportional	The fourth term of a proportion is called the fourth proportional to the other three terms.
	mean proportional	When the means of a proportion are the same, either of them is called the mean proportional between the other two.
	square root	The square root of a number is one of two equal factors of the number. The symbol for square root is "√".
	proportional line segments	Two lines are divided proportionally if the segments of one have the same ratio as the corresponding segments of the other.

REFERENCES	SUGGESTIONS
	A line segment is divided by the golden section if the ratio of the shorter section to the longer section is equal to the ratio of the longer section to the whole line segment. $\frac{a}{b} = \frac{b}{a + b}$ A line segment is said to be most harmoniously divided when it is divided into extreme and mean ratio by the golden section. Using these sections to make a rectangle, this rectangle is more pleasing to the eye than any other rectangle. The Greeks credit Pythagoras with the discovery of the golden section. Many famous painters have used the golden section in their work. Leonardo da Vinci, Michelangelo, Botticelli, and Dali are a few. Mother Nature uses the golden ratio in the design of the sunflower, the starfish, and others. Film: Donald in Mathmagic Land (26 min. - Color) Walt Disney Mr. Charles Jessen 237 W. Northwest Highway Park Ridge, Illinois An entertaining film--excellent material on applications of ratio and proportion, particularly the golden ratio.
A (341 - 344) C (208, 216) D (316 - 321) E (261 - 263)	Teacher note: The third proportional is the fourth term in a proportion having means which are the same number. Do not confuse third proportional with the third term.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PROPOSITIONS INVOLVING RATIO AND PROPORTION	
To develop the ability to discuss certain theorems and postulates involving ratio and proportion	Postulate: A line parallel to one side of a triangle and intersecting the other two sides divides the sides into proportional segments. Corol. On any two transversals, three parallel lines cut off segments, which when taken in the same order, have the same ratio. Th. The bisector of an interior angle of a triangle divides the opposite side into segments proportional to the adjacent sides. Postulate: If a line divides two sides of a triangle proportionally, it is parallel to the third side. Th. If two or more straight lines are cut by three or more parallel planes, their corresponding segments are proportional. Th. If a pyramid is cut by a plane parallel to the base and not passing through the vertex, the lateral edges and altitude are divided proportionally.

REFERENCES	SUGGESTIONS

A (330 - 333, 335, 347 - 354)

B (44 - 45, 116 - 120, selected exercises)

C (210 - 211, 214 - 215, 217 - 221)

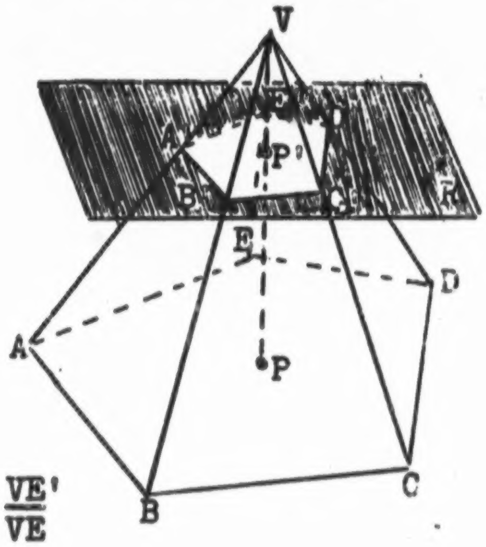
D (322 - 327)

E (268 - 272, 276 - 278)

G (83 - 85)

Have the pupil prepare a list of ways to prove line segments proportional.

Add this postulate to the summary of ways for proving lines parallel.



Given: Pyramid V-ABCDE,
altitude VP,
plane R parallel to the base and cutting the lateral edges at
A', B', C', D', and E'.

To prove: $\frac{VP'}{VP} = \frac{VA'}{VA} = \frac{VB'}{VB} = \frac{VC'}{VC} = \frac{VD'}{VD} = \frac{VE'}{VE}$

The complete theorem has three conclusions.

If a pyramid is cut by a plane parallel to the base and not passing through the vertex,

1. the lateral edges and altitude are divided proportionally
2. the section is a polygon similar to the base
3. the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.

The second conclusion will be discussed in the unit on similar polygons.

The third conclusion will be discussed in the unit on areas of polygons and circles.

The teacher should use good judgment in the selection of exercises from the references listed.

UNIT VIII

SIMILAR POLYGONS

14 Sessions

TOPICS AND OBJECTIVES

SIMILAR POLYGONS

To develop an understanding of similar polygons

To develop an understanding of theorems involving similar triangles

CONCEPTS, VOCABULARY, SYMBOLISM

similar polygons Similar polygons are polygons whose corresponding angles are equal and whose corresponding sides are in proportion.

The symbol for similar is "∼".

Th. (prove formally) Two triangles are similar if two angles of one are equal to two angles of the other.

Corol. Two triangles are similar if their corresponding sides are parallel.

Corol. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

Th. Two triangles are similar if an angle of one is equal to an angle of the other and the sides including these angles are in proportion.

Th. Two triangles are similar if their corresponding sides are in proportion.

Th. Corresponding altitudes, medians, and angle bisectors of similar triangles have the same ratio as any two corresponding sides.

Th. The ratio of the perimeters of two similar polygons is equal to the ratio of any pair of corresponding sides.

Th. If two polygons are composed of the same number of triangles, similar each to each and correspondingly placed, the polygons are similar.

Th. If a pyramid is cut by a plane parallel to the base and not passing through the vertex, the section formed is a polygon similar to the base.

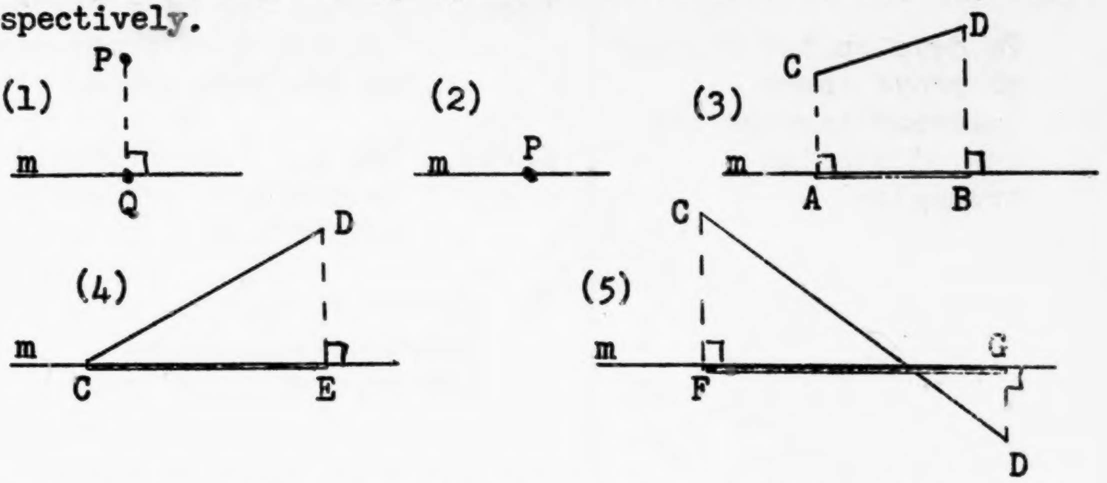
REFERENCES	SUGGESTIONS
(355 - 366, 390)	Note that neither condition **alone** is sufficient to insure that the two polygons are similar.
(116)	A square and a rectangle satisfy the condition that the corresponding angles are equal but the figures are not similar.
(223 - 238)	A square and a rhombus satisfy the condition that the corresponding sides are in proportion but the figures are not similar.
(328 - 333, 337 - 339)	
(291 - 303)	
(238 - 258)	
(84)	Film: Similar Triangles in Use (11 min. - Color) International Film Bureau 332 S. Michigan Avenue Chicago 4, Illinois Good story film--illustrates practical applications of similar triangles.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
PROJECTION To clarify the concept of projection with respect to points, lines, and planes	projection of a point on a line The projection of a point on a line is the foot of the perpendicular drawn from the point to the line. projection of a line segment on a line The projection of a line segment on a line is the segment included between the projection of the end points of the given line segment on the given line. projection of a point on a plane The projection of a point on a plane is the foot of the perpendicular drawn from the point to the plane. The perpendicular is called the projecting line. The plane is called the plane of projection. projection of a line segment on a plane The projection of a line segment on a plane is the segment whose end points are the projection of the end points of the given line segment on the plane. projection of a curve on a plane The projection of a curve on a plane is the projection of each point in the curve on the plane.

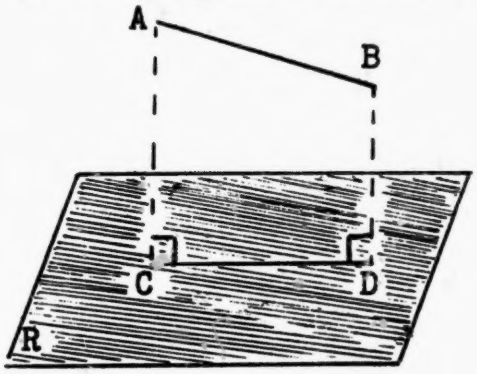
REFERENCES	
B	(54 - 58)
C	(108 - 109)
E	(307)
F	(259)
G	(81 - 83)

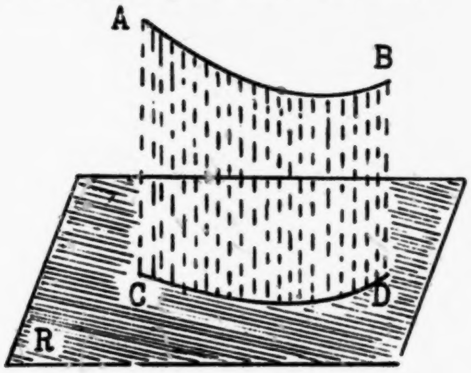
SUGGESTIONS

The projections referred to here are orthogonal projections. In the first drawing below, the projection of P on m is Q. In the second drawing P is its own projection on m. In the other drawings the projection of CD on m is AB, CE, and FG respectively.



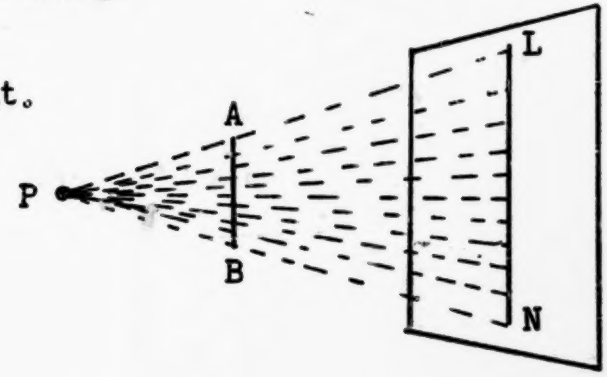
In the drawings below, R is the plane of projection, AB is the given line segment (or curve), and CD is the projection of AB.





A different type of projection, namely a central projection, is made by starting with a fixed point outside of the given point or line segment, and from the fixed point projecting lines through every point of the given line segment onto the plane of projection, as shown in the drawing.

P is the outside point.
AB is the given line segment.
LN is the projection.

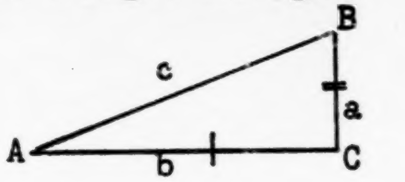


TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
RIGHT TRIANGLES	
To develop an understanding of theorems pertaining to the right triangle	Th. (prove formally) If the altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other.
To develop the ability to prove these theorems through the use of similar triangles	Corol. Either leg of a right triangle is the mean proportional between the hypotenuse and the projection of that leg on the hypotenuse.
	Corol. The altitude drawn to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.
	Th. (prove formally) In any right triangle, the square of the hypotenuse is equal to the sum of the square of the legs. (Pythagorean Theorem)

REFERENCES	SUGGESTIONS
A (374 - 389, 390 - 393)	**Problems pertaining to projections should be assigned to pupils upon completion of theorems concerning the right triangle.**
C (239 - 256)	
D (342 - 350)	
E (308 - 324, 330 - 338)	
F (260 - 290)	
Glenn, W. H. and Johnson, D. A. *The Pythagorean Theorem*.	The discovery of the proof of the "right triangle" theorem is credited to Pythagoras, a Greek mathematician and founder of a secret society. There are over 1,000 different proofs of this theorem.
Kline, Morris. *Mathematics in Western Culture*. (8-9, 32-41)	Among the authors of such proofs are Napoleon Bonaparte and President James A. Garfield. *The Pythagorean Theorem*, a book by E. S. Loomis, contains over 370 of these proofs. A good pupil project is "Other Proofs of the Pythagorean Theorem".

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. (prove formally) If the sum of the squares of two sides of a triangle is equal to the square of the third side, the triangle is a right triangle. This is the converse of the previous theorem.

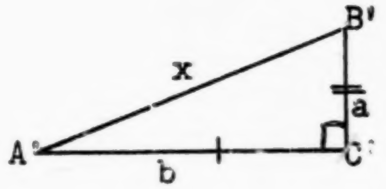
If the sum of the squares of two sides of a triangle is equal to the square of the third side, the triangle is a right triangle.



Given: $\triangle ABC$ with sides a, b, and c
and $c^2 = a^2 + b^2$

To prove: $\triangle$ ABC is a right triangle.

Plan: Draw a right triangle A'B'C' with legs equal to a and b and with hypotenuse x. (Right angle at C').



Proof

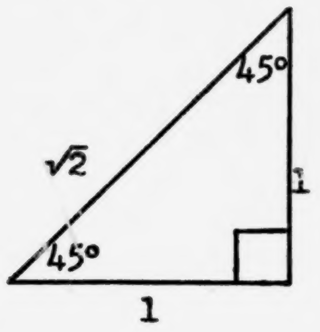
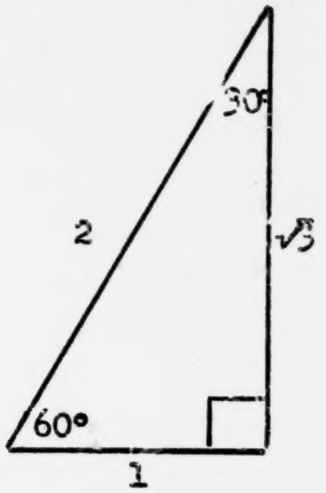
Statements	Reasons
1. $a^2 + b^2 = c^2$	1. Given
2. $a^2 + b^2 = x^2$	2. In a right triangle the square of the hypotenuse equals the sum of the squares of the two legs.
3. $c^2 = x^2$	3. Transitive axiom
4. $c = x$, $c > 0$	4. Powers and roots axiom
5. $\triangle ABC \cong \triangle A'B'C'$	5. s.s.s. = s.s.s.
6. $\angle C = \angle C'$	6. C.p.c.t.e.
7. $\angle C'$ is a right angle	7. Given
8. $\angle C$ is a right angle	8. Substitution axiom
9. $\therefore \triangle ABC$ is a right triangle	9. Definition of a right triangle

Corol. In any isosceles right triangle ($45° - 45° - 90°$) the ratio of the hypotenuse to either leg is $\sqrt{2}:1$.

Corol. In any $30° - 60° - 90°$ right triangle:

1. the ratio of the hypotenuse to the shorter leg is $2:1$.
2. the ratio of the hypotenuse to the longer leg is $2:\sqrt{3}$ (or $2\sqrt{3}:3$).
3. the ratio of the longer leg to the shorter leg is $\sqrt{3}:1$.

Th. The square of the diagonal of a rectangular prism is equal to the sum of the squares of the three dimensions.

REFERENCES	SUGGESTIONS
	Pupils should be given enough exercises pertaining to the 45-45-90 degree right triangle and the 30-60-90 degree right triangle so that the relationship between the sides becomes firmly established. Having pupils memorize the diagrams below will aid in the understanding and retention of these relationships. The continued recurrence of these specific relationships in more advanced mathematics courses amply justified requiring their memorization.  
B (93 - 95 selected exercises)	The proof of the "three-dimensional Pythagorean theorem" is simple and may be done as an exercise.
E (inserts between 312 and 313, 339)	
F (inserts between 272 and 273)	
	"The sum of the bases of any trapezoid is equal to zero!" is the title of an interesting fallacy problem which may be offered as a challenge to superior pupils. This is a difficult problem. The fallacy and its solution may be found in the appendix.

UNIT IX

CIRCLES AND SPHERES

22 Sessions

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM	
CIRCLES AND SPHERES		
To develop an understanding of the vocabulary pertaining to circles and spheres	circle	A circle is the set of points in a plane which are equally distant from a fixed point in the plane called the center. A circle separates the set of points in a plane into three disjoint subsets: the circle itself, the interior of the circle, and the exterior of the circle. The symbol for circle is "⊙".
	interior of a circle	The interior of a circle is the set of all the points in the plane of the circle whose distance from the center is less than the radius.
	exterior of a circle	The exterior of a circle is the set of all the points in the plane of the circle whose distance from the center is greater than the radius.
	radius (plural radii)	A radius of a circle is a line segment from the center of a circle to any point in the circle.
	diameter	The diameter of a circle is a line segment which passes through the center of the circle and whose end points are in the circle.
	arc	An arc is the union of two points in a circle and all the points in the circle between them. An arc is a subset of a circle. The symbol for arc is "⌒". An arc is named by its end points.
	semicircle	A semicircle is an arc which is half of a circle.
	minor arc	A minor arc is an arc less than a semicircle.
	major arc	A major arc is an arc greater than a semicircle.

REFERENCES

A (2, 3, 206 - 209, 225, 235, 314)

B (144 - 154 selected exercises)

C (351 - 352, 452)

D (209 - 215)

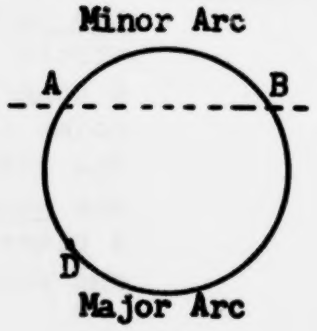
E (14 - 15, 341 - 343)

F (68 - 70, 71 - 79 selected exercises, 322 - 327)

G (53)

SUGGESTIONS

To avoid confusion when naming minor arcs or major arcs, use at least three letters to name major arcs.

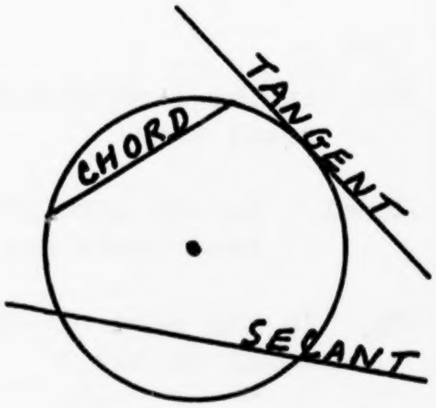


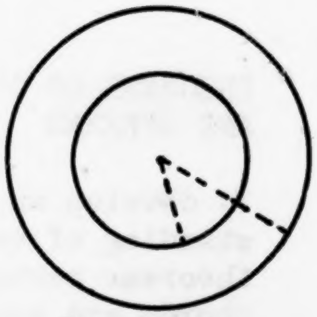
$\overset{\frown}{AB}$ is the minor arc.

$\overset{\frown}{ADB}$ is the major arc.

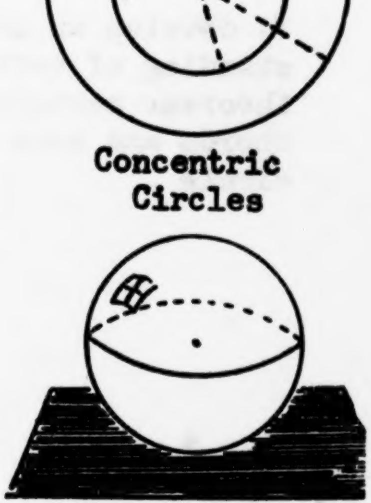
sphere	A sphere is the set of all points in space which are equally distant from a fixed point called the center. A sphere separates all the points of space into three subsets: all the points in the sphere, all the points in the interior of the sphere, and all the points in the exterior of the sphere.
radius	A radius of a sphere is a line segment from the center of the sphere to any point in the sphere.
diameter	A diameter of a sphere is a line segment which passes through the center of the sphere and whose end points are in the sphere.
hemisphere	A hemisphere is half a sphere.
Postulate:	The diameter bisects the circle and conversely.
Postulate:	A straight line cannot intersect a circle or a sphere in more than two points.
equal circles (spheres)	Equal circles (spheres) are circles (spheres) having equal radii or equal diameters. All radii and all diameters of the same or equal circles (spheres) are equal.
concentric circles (spheres)	Concentric circles are circles in the same plane with the same center and with unequal radii. Spheres are concentric if they have the same center and unequal radii.
chord	A chord is a line segment connecting any two points in a circle (sphere).
secant	A secant is a line which intersects a circle (sphere) in two points.
tangent	A tangent to a circle is a line which is coplanar with the circle and has only one point in common with the circle. A tangent to a sphere is a line which has only one point in common with the sphere. The common point is called the point of tangency or the point of contact. A plane is tangent to a sphere if it has one and only one point in common with the sphere.

REFERENCES	SUGGESTIONS
	Note that the sphere is the surface, not the portion of space enclosed by the surface.


CHORD
TANGENT
SECANT



Concentric Circles



Plane Tangent to a Sphere

line of centers The line of centers of two coplanar circles (two spheres) is the line segment joining the centers of the circles (spheres).

central angle The central angle of a circle (sphere) is an angle whose vertex is the center of the circle (sphere) and whose sides are radii of the circle (sphere).

equal arcs Equal arcs are arcs in the same or equal circles which subtend equal central angles.

Since a definition is reversible, this means that in the same or equal circles equal arcs subtend equal central angles and equal central angles intercept equal arcs.

midpoint of an arc The midpoint of an arc is the point in the arc which divides it into two equal arcs.

unequal arcs Unequal arcs in the same or equal circles subtend unequal central angles, the longer arc subtending the greater central angle.

Since a definition is reversible, this means that in the same or equal circles unequal minor arcs subtend unequal central angles of the same order and unequal central angles intercept unequal minor arcs of the same order.

THEOREMS ON CIRCLES AND SPHERES

To develop an understanding of certain theorems pertaining to chords and arcs of a circle

Th. In the same circle or in equal circles, equal chords have equal arcs.

Corol. In the same circle or in equal circles, the longer of two chords has the longer minor arc.

Th. In the same circle or in equal circles, equal arcs have equal chords.

Corol. In the same circle or in equal circles, the longer of two minor arcs has the longer chord.

Th. If a line through the center of a circle is perpendicular to a chord, it bisects the chord and its arc.

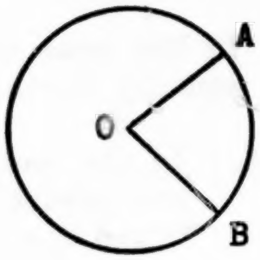
Corol. A line through the center of a circle and bisecting a chord (not a diameter) is perpendicular to the chord.

REFERENCES

A	(209 - 213, 315 - 316)
C	(353 - 356)
D	(215 - 219, 223 - 224)
E	(344 - 350, 498 - 501)
F	(340 - 344)

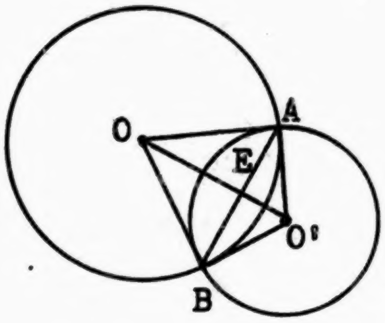
SUGGESTIONS

The central angle AOB is said to intercept the arc AB.
The arc AB is said to subtend the central angle AOB.



Pupils should periodically add to their lists of ways to prove line segments equal, ways to prove angles equal, and others. They should now begin a list of ways to prove arcs equal.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. The perpendicular bisector of a chord of a circle passes through the center of the circle.
	Th. The perpendicular bisectors of two non-parallel chords of a circle intersect at the center of the circle.
	Th. In the same circle or in equal circles, equal chords are equally distant from the center.
	Corol. In the same circle or in equal circles, unequal chords are unequally distant from the center, the longer chord being the nearer.
	Th. In the same circle or in equal circles, chords equally distant from the center are equal.
	Corol. In the same circle or in equal circles, chords unequally distant from the center are unequal, the chord nearer to the center being the longer.
	Th. If two circles intersect, the line of centers is the perpendicular bisector of the common chord.

REFERENCES	SUGGESTIONS
	The theorem concerning the perpendicular bisectors of two nonparallel chords may be proved as an exercise. In the unit on constructions this theorem may be used to find the center of a circle.
	This theorem may be proved as an exercise. Use congruent triangles. 

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
To develop an understanding of certain theorems involving tangents to circles and spheres	Th. (prove formally) If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.
	Corol. A straight line perpendicular to a radius at its outer extremity is tangent to the circle. (Line and circle are coplanar.)
	Corol. A line coplanar to a circle and perpendicular to a tangent at the point of tangency passes through the center of the circle.
	Corol. A line from the center of a circle and perpendicular to a tangent passes through the point of tangency.
	Th. If a plane is tangent to a sphere, it is perpendicular to the radius drawn to the point of tangency.

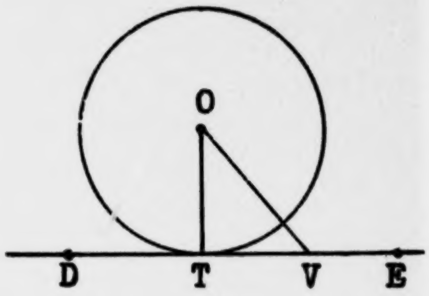
REFERENCES	
A	(214 - 220, 221 - 223 selected exercises)
B	(149 - 151)
C	(356 - 362)
D	(224 - 233)
E	(350 - 356)
F	(70)

SUGGESTIONS

Given: DE tangent to circle O at T.
OT is a radius.

To prove: $DE \perp OT$

Plan: Draw a line OV to any point V distinct from T on line DE.
Prove $OV > OT$.



Proof

Statements	Reasons
1. DE tangent to circle O at T	1. Given
2. OV, from point V on DE and distinct from T	2. Construction
3. V is in the exterior of circle O.	3. V is not in the circle since a tangent has only one point in common with a circle and V is distinct from T. V is not in the interior of the circle since a line joining V and T would intersect the circle in two points.
4. $OV > OT$	4. Definition of the exterior of the circle.
5. $\therefore OT \perp DE$	5. The shortest distance from a point to a line is a perpendicular.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

length of a tangent to a circle from an external point The length of a tangent to a circle from an external point is the length of a segment joining the external point to the point of tangency.

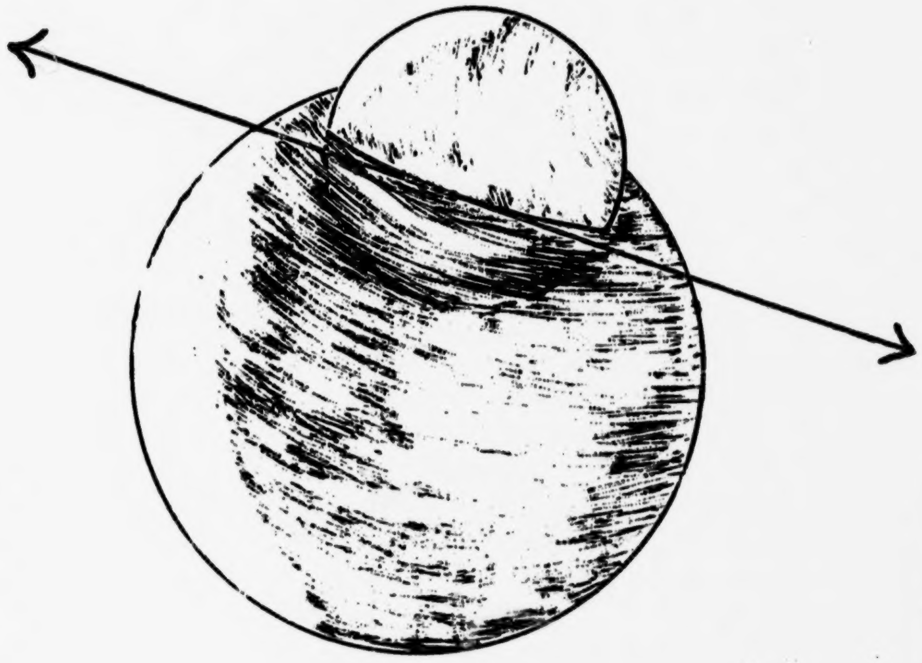
Th. The tangents to a circle from an external point are equal.

Corol. If two tangents are drawn to a circle from an external point, they make equal angles with a line segment joining the point to the center of the circle.

tangent circles Two circles are tangent to each other if they are coplanar and tangent to the same line at the same point.

tangent spheres Two spheres are tangent to each other if they are both tangent to the same plane at the same point.

REFERENCES	SUGGESTIONS
	Note: Two spheres are not necessarily tangent to each other if they are each tangent to the same line at the same point. See drawing. Two spheres can be tangent to the same line at the same point and intersect in which case the spheres are not tangent to each other.



internally tangent circles and spheres Spheres and coplanar circles are internally tangent if they are tangent and if one lies wholly within the other.

externally tangent circles and spheres Spheres and coplanar circles are externally tangent if they are tangent and if one lies wholly outside the other.

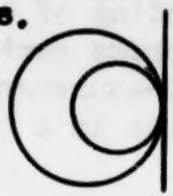
common tangent A common tangent is a line tangent to each of two coplanar circles.

If the common tangent intersects the line of centers, then it is a common internal tangent. If the common tangent does not intersect the line of centers, then it is a common external tangent.

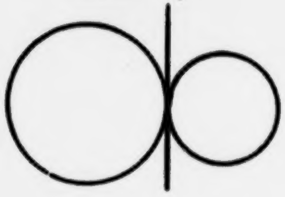
Th. If two circles are tangent to each other, their line of centers passes through the point of tangency.

Do not allow pupils to confuse internally and externally tangent circles with common internal and external tangents.

These circles are internally tangent.
The line is a common external tangent.

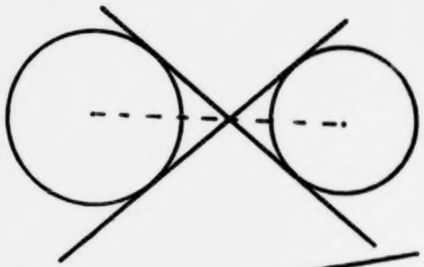


These circles are externally tangent.
The line is a common internal tangent.

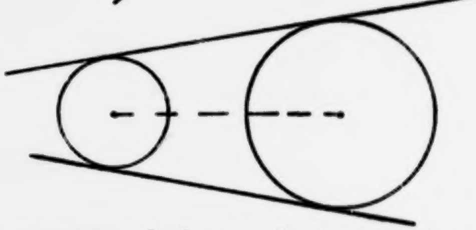


A line is a common internal tangent to two circles if the circles lie on opposite sides of the line.
A line is a common external tangent to two circles if the circles lie on the same side of the line.

These lines are common internal tangents.



These lines are common external tangents.



Have the pupils discover how many common internal tangents or common external tangents the following have.

1. two concentric circles
2. two internally tangent circles
3. two externally tangent circles
4. two coplanar circles that intersect
5. two coplanar circles that do not intersect nor contain each other

Determine the maximum number of common internal and external tangents.

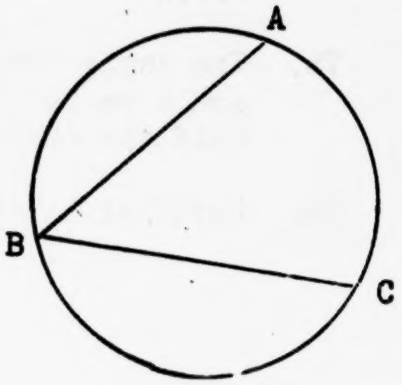
TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
To develop an understanding of certain theorems pertaining to measurement of angles in a circle	Postulate: A central angle has the same number of degrees as its intercepted arc.
	inscribed angle An inscribed angle is an angle formed by two chords drawn from the same point in a circle. An inscribed angle is said to intercept the arc between its sides An angle is said to be inscribed in an arc if its vertex is in the arc and its sides terminate in the end points of the arc.
	Th. (prove formally) An inscribed angle is measured by half the intercepted arc.
	Corol. An angle inscribed in a semicircle is a right angle.
	Corol. In the same or in equal circles if two inscribed angles intercept the same or equal arcs, the angles are equal.
	Corol. The circle whose diameter is the hypotenuse of a right triangle passes through the vertex of the right angle of the triangle.
	Corol. The opposite angles of an inscribed quadrilateral are supplementary.
	inscribed polygon An inscribed polygon is a polygon whose vertices lie in the circle.
	circumscribed polygon A circumscribed polygon is a polygon whose sides are tangent to the circle.

REFERENCES	
A	(230 - 258, 260 - 262)
C	(363 - 373)
D	(234 - 246)
E	(380 - 392)
F	(327 - 340)

SUGGESTIONS

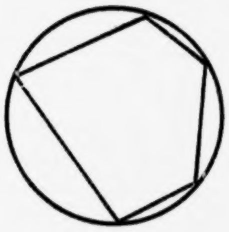
It is advisable to review the necessary elementary algebra in order to prove theorems and exercises involving angle measurement.

Inscribed $\angle ABC$ intercepts $\widehat{AC}$ and is said to be inscribed in $\widehat{ABC}$.

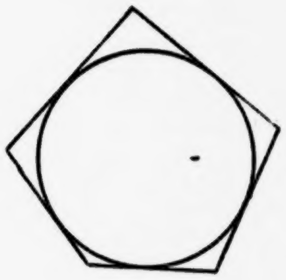


Have pupils note that an angle inscribed in a minor arc is obtuse and an angle inscribed in a major arc is acute.

The polygon is inscribed in the circle.
The circle is circumscribed around (circumscribes) the polygon.



The polygon is circumscribed about the circle.
The circle is inscribed in the polygon.



Th. An angle formed by two chords intersecting within a circle is measured by half the sum of its intercepted arcs.

Th. An angle formed by a tangent and a chord drawn from the point of tangency is measured by half of its intercepted arc.

Th. The angle between two secants, two tangents, or a tangent and a secant intersecting outside a circle is measured by half the difference of their intercepted arcs.

Th. Parallel lines intercept equal arcs in a circle.

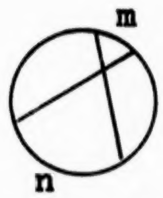
REFERENCES	SUGGESTIONS

To help pupils remember these theorems state that:

1. if the vertex of the angle formed is within the circle, it is measured by half the sum of the intercepted arcs.
2. if the vertex of the angle formed is in the circle, it is measured by half the intercepted arc.
3. if the vertex of the angle formed is outside the circle, the angle is measured by half the difference of the intercepted arcs.

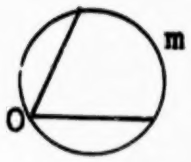
Some texts generalize these four theorems as "The angle formed by two intersecting lines, either cutting or tangent to a circle, is measured by half the sum of the intercepted arcs." To apply this theorem, it is necessary to know that:

1. if the intercepted arc is concave when viewed from the vertex, it is called a positive arc.
2. if the arc is convex when viewed from the vertex, it is called a negative arc.



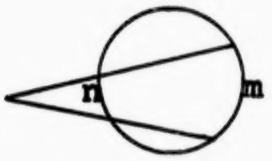
$\widehat{m}$ is concave

$\widehat{n}$ is concave



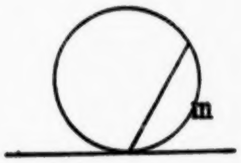
$\widehat{m}$ is concave

$\widehat{n}$ is zero



$\widehat{m}$ is concave

$\widehat{n}$ is convex



$\widehat{m}$ is concave

$\widehat{n}$ is zero

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
To develop an understanding of certain theorems pertaining to spheres	**Th.** The intersection of a plane and a sphere is a circle. **axis of a circle of a sphere** The axis of a circle of a sphere is the diameter of the sphere perpendicular to the plane of the circle. **great circle of a sphere** The great circle of a sphere is the intersection of the sphere and a plane that passes through the center of the sphere. **small circle of a sphere** The small circle of a sphere is the intersection of the sphere and a plane that does not pass through the center of the sphere. **Corol.** The axis of a circle of a sphere passes through the center of the circle. **Corol.** All great circles of the same or equal spheres are equal.

REFERENCES	
A	(225, 259)
B	(144 - 149)
C	(454)
D	(220 - 223)
E	(362 - 364)
F	(74 - 76)
G	(54 - 56)

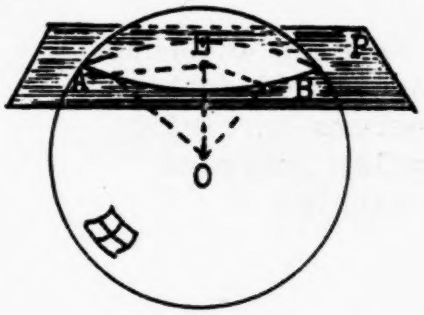
SUGGESTIONS

Prove this theorem as an exercise.

Given: Sphere O intersected by plane P.

To prove: The section formed is a circle.

Plan: Take any two points, A and B, in the intersection of the plane and the sphere.
Draw OE perpendicular to P.
Draw OA, OB, EA, and EB.



Proof

Statements	Reasons
1. Sphere O intersected by plane P	1. Given
2. OE $\perp$ P	2. So constructed
3. OA = OB	3. Radii of the same sphere are equal.
4. OE = OE	4. Identity axiom
5. $\angle$OEA and $\angle$OEB are right angles.	5. Perpendicular lines form right angles.
6. $\triangle$ OEA and $\triangle$ OEB are right triangles	6. Definition of right triangle
7. $\triangle$ OEA $\cong$ $\triangle$ OEB	7. rt. $\triangle$s h.s. = h.s.
8. EA = EB	8. C.p.c.t.e.
9. $\therefore$ The section is a circle with E as the center.	9. Definition of a circle. A and B are any points in the intersection.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Corol. Three points in a sphere determine a circle.
	Corol. A great circle bisects a sphere.
	Th. The intersection of two spheres is a circle.
To develop an understanding of certain theorems involving similar polygons in circles	**Th.** (prove formally) If two chords intersect within a circle the product of the segments of one is equal to the product of the segments of the other.

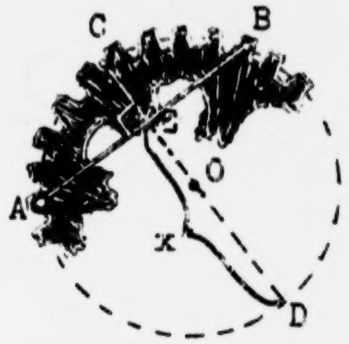
REFERENCES	
A	(367 - 373)
C	(374 - 378)
D	(334 - 336)
E	(396 - 402)
F	(345 - 350)

SUGGESTIONS

Note that the product of the line segments means the product of the measures of the line segments.

Pupils often ask about practical applications of geometric concepts.
Here is an excellent example of a practical application.

This problem might occur in any factory using machines.
A gear wheel is broken during operation and only a fragment remains.
A new wheel must be made immediately.
The diameter of the original gear is unknown.
Can the diameter of the original gear be found using only the remaining fragment?



SOLUTION: A and B are two points on the circular arc portion of the gear.
Segment AB is measured and found to be 9 inches.
E is the midpoint of AB.
From E a perpendicular line is drawn intersecting AB at C (the midpoint of arc AB).
CE is measured and found to be 3 inches.
The diameter CD is found as follows.
Let ED = x inches.
Since CE ∘ ED = AE ∘ EB, then

$$3 \circ x = 4\frac{1}{2} \circ 4\frac{1}{2}$$

$$3x = \frac{81}{4}$$

$$x = 6\frac{3}{4}$$

Therefore, the diameter of the wheel is $9\frac{3}{4}$ inches.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. If from a point outside a circle two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.
	Th. If from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment; or the product of the secant and its external segment equals the square of the tangent.

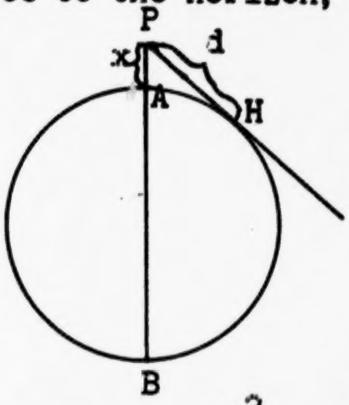
A review of quadratic equations is advisable at this time.

A practical application of this theorem may be illustrated by the following problem:

If one were standing at a point above the earth's surface, for example at the top of a lighthouse or a tower, and one were x feet up in the air, how far could one see to the horizon?

The radius of the earth equals approximately 3,960 miles.

Let PH, the distance one could see to the horizon, be "d" miles.



$$d^2 = PA \cdot PB$$

$$d^2 = \frac{x}{5280}\left(\frac{x}{5280} + 7920\right)$$

$$d^2 = \left(\frac{x}{5280}\right)^2 + \frac{7920x}{5280}$$

$$d^2 = \left(\frac{x}{5280}\right)^2 + \frac{3x}{2}$$

If x is relatively small, the quantity $\left(\frac{x}{5280}\right)^2$ will be small enough to be negligible. The following formula is a close approximation:

$$d^2 = \frac{3x}{2} \quad \text{or } d = \sqrt{\frac{3x}{2}}$$

where d is the distance in miles to the horizon one can see when one is x feet above the surface of the earth!

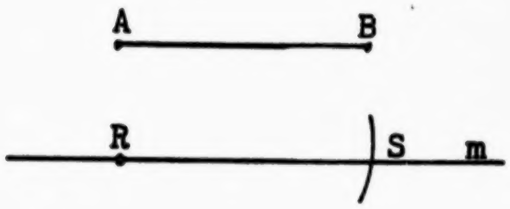
UNIT X

GEOMETRIC CONSTRUCTIONS

5 Sessions

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
FUNDAMENTAL CONSTRUCTIONS	
To develop the ability to construct geometric figures	**drawing** A drawing is a representation on paper using a protractor, marked rule, compass, straightedge, or any other desired drawing instrument.
	construction A construction is a drawing using only a compass and straightedge. A straightedge only can be used to draw lines. A compass can be used to mark off equal segments and to construct circles or arcs of circles. There are four steps to be followed in a construction problem. 1. State the given. 2. State what is required. 3. State the method of construction. 4. Prove the construction.
	Construction #1 Construct a line segment equal to a given line segment.

REFERENCES	SUGGESTIONS
F (359) Norton, M. Scott. Geometric Constructions	Neither a protractor nor the marks on a rule may be used in any construction problem.
A (4, 6)	#1 Given: Line segment AB



Required: To construct segment RS equal to AB.

Method: Draw any line m and mark any point R on it.
Set compass with one point at A and the other at B.
Without changing the setting, place one point of the compass at R and mark an arc cutting m in S.

Proof

Statement	Reason
1. AB = RS	1. Radii of the same or equal circles are equal.

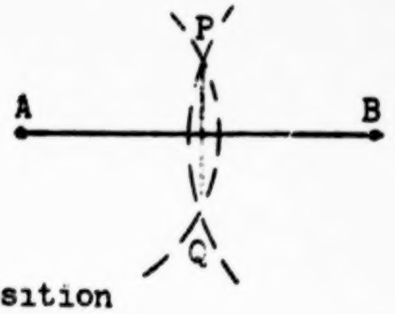
TOPICS AND OBJECTIVES		CONCEPTS, VOCABULARY, SYMBOLISM
	Construction #2	Construct the perpendicular bisector of a given line segment. This construction is also satisfactory for bisecting a given line segment.

REFERENCES

SUGGESTIONS

A (5, 27 - 28)

#2 **Given:** Line segment AB



Required: To construct the perpendicular bisector of AB

Method: Open the compass to any position greater than one-half AB.
With one point of the compass at A, describe an arc above and below AB.
With the same setting, place one point of the compass at B and describe an arc above and below AB intersecting the first arcs at P and Q.
Connect P and Q.

Proof

Statements	Reasons
1. AP = BP	1. All radii of equal circles are equal.
2. AQ = BQ	2. All radii of equal circles are equal.
3. ∴ PQ is the perpendicular bisector of AB.	3. Two points each equally distant from the end points of a given line segment determine the perpendicular bisector of the line segment.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Construction #3 To construct an angle equal to a given angle.
	Construction #4 To bisect a given angle.
	Construction #5 To construct a line perpendicular to a given line at a given point in the line.
	Construction #6 To construct a line perpendicular to a given line from a given point not in the line.

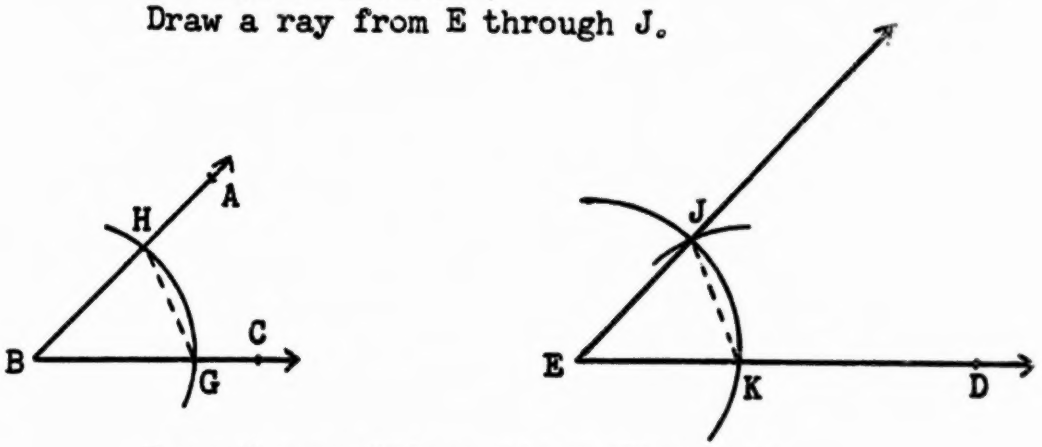
REFERENCES | SUGGESTIONS

A (19, 118)

#3 Given: $\angle ABC$

Required: To construct an angle equal to $\angle ABC$

Method: Draw ray ED.
Place one point of the compass at B and with any convenient setting describe an arc intersecting the sides of the angle at G and H.
With the same setting, place one point of the compass at E and describe an arc intersecting ED at K.
Place one point of the compass at G and the other point at H.
With the same setting place one point of the compass at K and describe an arc intersecting the other arc at J.
Draw a ray from E through J.



Draw the auxiliary lines GH and KJ.

Proof

Statements	Reasons
1. BG = EK	1. All radii of equal circles are equal.
2. BH = EJ	2. Same reason as 1.
3. GH = KJ	3. Same reason as 1.
4. $\triangle GBH \cong \triangle KEJ$	4. s.s.s. = s.s.s.
5. $\angle ABC = \angle JED$	5. C.p.c.t.e.

A (18, 117)

A (24, 118)

A (25, 119)

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Construction #7 To construct two triangles congruent by means of: a.s.a. = a.s.a. s.s.s. = s.s.s. s.a.s. = s.a.s. Since triangle congruency was postulated no formal proof for these constructions is necessary.
	Construction #8 To construct a line parallel to a given line at a given distance from the given line.
	Construction #9 To construct a line parallel to a given line through a given point not in the given line.
	Construction #10 To divide a line segment into any number of equal parts.

REFERENCES

A (65 - 66)

A (140)

A (145)

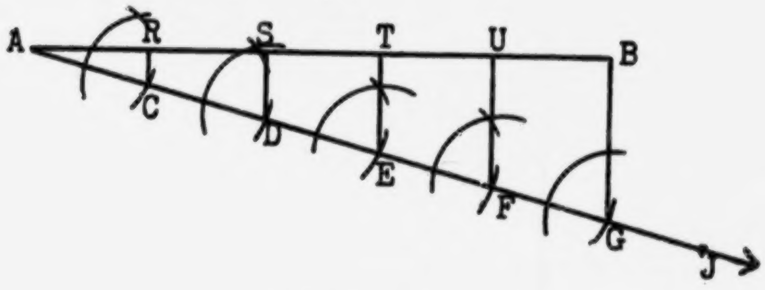
A (191)

SUGGESTIONS

#10 Given: Line segment AB

Required: To divide AB into any number of equal parts. (For example, 5)

Method: Draw any ray AJ at a convenient angle with AB.
With the compass at A, and any convenient setting, mark off five arcs in succession on AJ so that AC = CD = DE = EF = FG.
Draw BG.
Construct lines parallel to BG (by means of equal corresponding angles) through F, E, D, and C.
These parallels intersect AB in U, T, S, and R respectively.



Proof

Statements	Reasons
1. AC = CD = DE = EF = FG	1. All radii of equal circles are equal.
2. BG ‖ FU ‖ ET ‖ DS ‖ CR	2. If two straight lines are cut by a transversal so that the corresponding angles are equal, the lines are parallel.
3. ∴ BU = UT = TS = SR = RA	3. If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on every transversal.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Construction #11 To construct two tangents to a given circle from a given external point.

REFERENCES	SUGGESTIONS

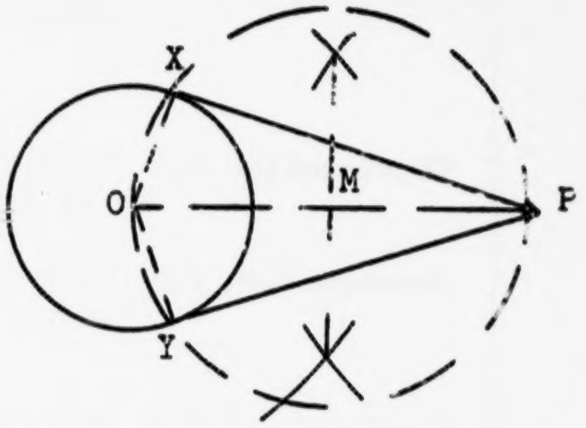
A (476)

F (369)

#11 Given: Circle O with external point P.

Required: To construct tangents to circle O from P

Method: Draw OP.
Bisect OP, the midpoint of which is M.
With M as a center and MO as a radius, construct a circle intersecting circle O at X and Y.
Draw PX and PY.



Draw auxiliary lines OX and OY.

Proof

Statements	Reasons
1. OM = MP	1. Definition of the bisector of a line segment
2. Circle with center at M and radius OM passes through P	2. Definition of a circle
3. OP is a diameter.	3. Definition of a diameter
4. $\overset{\frown}{OXP}$ and $\overset{\frown}{OYP}$ are semicircles.	4. A diameter bisects a circle.
5. ∠OXP and ∠OYP are right angles.	5. Angles inscribed in a semicircle are right angles.
6. ∴ PX and PY are tangent to circle O.	6. If a line is perpendicular to a radius at its outer extremity, the line is tangent to the circle.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Construction #12 To construct a circle circumscribing a given triangle. This construction may also be used to determine a circle when given three non-collinear points.
	Construction #13 To inscribe a circle within a given triangle.
	Construction #14 To construct the fourth proportional to three given line segments.
	Construction #15 To divide a line segment into parts that have the same ratio as two given line segments.
	Construction #16 To construct the mean proportional between two given line segments.
	Construction #17 To inscribe a regular hexagon in a circle. This construction is similar to the construction used when inscribing an equilateral triangle in a circle.
	Construction #18 (optional) To inscribe a square in a circle.
	Construction #19 (optional) To construct a circle through nine points, three of which are the midpoints of the sides of a given triangle, three of which are the feet of the altitudes of the same triangle, and three of which are the midpoints of the three segments from the orthocenter to the vertices of the same triangle.
	Construction #20 (optional) To transform a polygon of any number of sides into a triangle equal in area.
	Construction #21 (optional) To transform a rectangle into a square equal in area.

REFERENCES

A (286)
F (369)
A (287)
F (370)
A (345)
F (372)
A (346)
A (375)
F (373)
A (442)
A (442)
C (491)
A (429)
A (430)

SUGGESTIONS

The proofs of constructions #12 and #13 are dependent upon locus theorems and should be omitted at this time.

A good project for pupils is the "Three Famous Construction Problems" that cannot be solved by the use of the straightedge and the compass alone.
These constructions are:
1. the trisection of an angle,
2. the duplication of a cube,
3. the squaring of a circle.

In another sense, these problems have been solved. The solutions, algebraic in nature, proved that the above three constructions cannot be accomplished using a compass and straightedge alone.

Despite the fact that these problems have been proved impossible to solve, periodically someone comes up with a "solution" to one or more of these problems.
The fact that these solutions are published and appear in print lends them a certain sense of undeserved authenticity.
Some of these so-called solutions have even appeared in the Congressional Record.
The errors in these solutions generally fall into one of the following categories:
1. The solution is based on false assumptions.
2. The solution violates a rule that a straightedge can be used for drawing a line through two known points but cannot be used for anything else.
3. In the case of the trisection of an angle, the construction will work for certain angles but not for all angles.
4. The constructions give approximations of the requirements--sometimes very close approximations--but do not fulfill the requirement exactly.

In recent years, most of the attempts at solution have been guilty of this last error.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Construction #22 (optional) To divide a line segment into the golden ratio.

REFERENCES

C (493)

D (319 - 320)

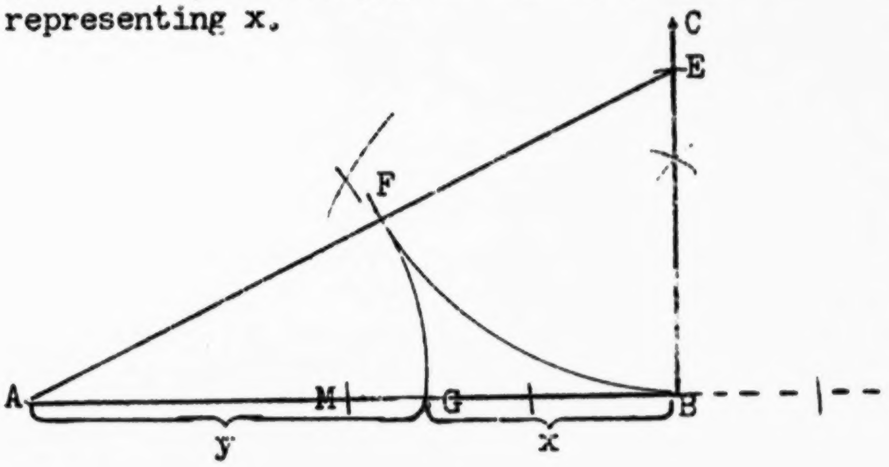
Cundy, H. M. and
Rollett, A. P.
Mathematical Models.
(68-69)

SUGGESTIONS

#22 Given: Line segment AB.

Required: To divide AB into two parts, x and y, such that $\frac{x}{y} = \frac{y}{x + y}$, i.e., to divide AB into the golden ratio.

Method: Bisect AB. Let M be the midpoint of AB. At B construct BC perpendicular to AB. On BC, locate point E so that EB = MB. Draw AE. With E as a center and EB as a radius, describe an arc intersecting AE at F. EF = EB. With A as a center and AF as a radius, describe an arc intersecting AB at G. G divides AB into the golden ratio with AG representing y in the ratio and GB representing x.



Outline of proof:

Let x equal the shorter segment and y equal the longer segment. Let the length of the given line segment, x + y, equal 1.

(1) By the golden ratio, $\frac{x}{y} = \frac{y}{1}$

(2) then $y^2 = x$

But since $x + y = 1$,

(3) $x = 1 - y$

(4) Substituting in (2), $y^2 = 1 - y$, $y^2 + y - 1 = 0$.

(5) Solving by the quadratic formula, $y = \frac{-1 \pm \sqrt{5}}{2}$

(6) Discarding the negative root, $y = \frac{-1 + \sqrt{5}}{2}$, $y = .618+$

In the construction, let AB = 1. Then $AM = MB = \frac{1}{2}$. $BE = \frac{1}{2}$. Since $\triangle$ ABE is a right triangle, by the Pythagorean Theorem $AE = \frac{\sqrt{5}}{2}$. $FE = EB = \frac{1}{2}$, then $AF = \frac{\sqrt{5}}{2} - \frac{1}{2}$ $AF = \frac{\sqrt{5} - 1}{2}$

Finally, since AG = AF, $AG = \frac{\sqrt{5} - 1}{2}$. AG = .618+

UNIT XI

LOCUS

9 Sessions

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
LOCUS To develop an understanding of the meaning of locus	locus A locus (plural, loci) is a set of points and only those points which satisfy one or more given conditions. In geometry, this set of points takes the form of geometric figures such as points, lines, planes, and solids. In algebra, this set of points takes the form of the graph of an equation. The geometric figure or graph contains all the points which satisfy the given conditions and no points which do not satisfy the conditions.
	solution of locus exercises A solution of a locus exercise should consist of two parts: 1. A drawing in which the locus is clearly seen. Two colors may be used--one color for the given conditions and the other for the locus. 2. An accurate description of the locus beginning with the words, "The locus is . . . ."

REFERENCES	
A	(263 - 268)
C	(333 - 340)
D	(279 - 282)
E	(459 - 462)
F	(375 - 379)

SUGGESTIONS

The concept of locus may be presented as the path taken by a moving point.
This is an extremely important concept.
The literal translation of the word locus, viz., "place" is of little help in the understanding.
The correct language involving the use of the word is hard to understand.
If pupils have plenty of experience determining loci under given simple conditions, difficult problems will appear less complex.
If the teacher is aware of the trouble spots in the unit, he can, with skillful direction and assistance, overcome much of the difficulty experienced by pupils in understanding locus.
For example, how can the following statement be expressed in words easier for pupils to understand? "The locus of a point which is the vertex of the right angle of a right triangle with a fixed hypotenuse is a circle with the hypotenuse as the diameter."

Suggested method I

If we are given a fixed line segment which is the hypotenuse of a right triangle, the path taken by a point which moves so that it is always the vertex of the right angle of this right triangle, will be a circle with the fixed line as the diameter.

Suggested method II

Start with a fixed line segment and call it the hypotenuse of a right triangle.
Using this line as the hypotenuse construct a number of right triangles.
The locus of the vertices of these right triangles is a circle with the fixed hypotenuse as the diameter.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	description of a locus In order to describe the locus determined by certain conditions, state: 1. the class of geometric figures to which the locus belongs, 2. specific information about the location of the geometric figure.

REFERENCES	SUGGESTIONS

Samples of locus problems:

1. Find the locus of points (in a plane) two inches from a given point P.
2. Find the locus of points six centimeters from line m.
3. Find the locus of points which are one foot from a given point O and which are three inches from a given line m. Line m is one inch from O.
4. Find the locus of points in space which are four inches from a given line segment MN.

Solutions:

Examples of description of a locus: "The locus is . . .

	The class of geometric figures to which the locus belongs	Specific information concerning the location of the geometric figure
1.	.. a circle	with center at P and radius of two inches."
2.	.. two lines	both parallel to line m and with one line on either side six centimeters away."
3.	.. four points	which are the intersection of a circle and two parallel lines. The circle has a center at O and a radius of one foot. The two parallel lines are each parallel to m with one line on either side three inches away from m."
4.	.. a cylindrical surface and two hemispheres	whose axis is line MN and whose radius is 4 inches, whose centers are M and N and whose radii are 4 inches."

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
LOCI IN A PLANE	
To develop an understanding of theorems involving locus	**Postulate:** In a plane, the locus of points at a given distance from a given point is a circle whose center is the given point and whose radius is the given distance. **Postulate:** In a plane, the locus of points equidistant from two parallel lines is the line midway between them and parallel to each of them. **Postulate:** In a plane, the locus of points at a given distance from a given line is a pair of lines parallel to the given line and at the given distance from the line. **Th.** In a plane, the locus of points equally distant from two given points is the perpendicular bisector of the line segment joining the two points. **Corol.** In a plane, the locus of points equidistant from two intersecting lines is the pair of perpendicular lines bisecting the angles formed by the lines. **Th.** In a plane, the locus of points equally distant from the sides of an angle is the bisector of the angle. **Th.** In a plane, the locus of the vertex of the right angle of a right triangle with a fixed hypotenuse is a circle whose diameter is the hypotenuse.
To develop the ability to visualize compound loci	intersection of loci When a set of points must satisfy two or more given conditions, then the locus is the intersection of the loci of the individual given conditions. concurrent lines Concurrent lines are three or more lines having one and only one point in common.

REFERENCES	SUGGESTIONS
A (271 - 276) C (341 - 344) D (282 - 283) E (463 - 467)	Pupils will find it helpful in solving locus exercises to follow a definite procedure. 1. Decide what is fixed in position and make a drawing. 2. Decide what is variable. 3. Locate several points (variables) that satisfy the given conditions. Be sure that there is a sufficient number of points close enough together so that a general trend can be clearly seen. 4. Complete the locus by considering any special position of the variable; e.g. the end points of a line segment. In the proof of locus theorems, have pupils prove two sets of points are the same. 1. Every point is an element of the set of points that satisfy the given conditions. 2. Every point that satisfies the given conditions is a member of the set of points.
A (278 - 280) C (345 - 350) D (289 - 294) E (467 - 469) F (379 - 381)	Since locus is defined as a set of points, the locus can be a null set, a finite set, or an infinite set. When determining the intersection of two loci: 1. construct the locus that satisfies the first condition. Call this Set A. 2. construct the locus that satisfies the second condition. Call this Set B. 3. determine the set of points that satisfy both conditions. Call this $A \cap B$. If the two loci do not intersect, then $A \cap B = \emptyset$.
A (286 - 295) D (298 - 305) E (469 - 476)	

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
To develop an understanding of theorems involving compound loci	Th. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices. The point is the center of the circle that circumscribes the triangle and is called the circumcenter of the triangle. Th. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides. The point is the center of the circle inscribed within the triangle and is called the incenter of the triangle. Th. The altitudes of a triangle are concurrent. The point is called the orthocenter of the triangle. Th. The medians of a triangle are concurrent in a point which is two-thirds the distance from a vertex to the midpoint of the opposite side. The point is called the centroid of the triangle. The centroid of any plane figure is also the center of gravity. A triangle or any plane figure suspended at its centroid will hang horizontally in space.
LOCI IN SPACE	
To develop an understanding of theorems involving loci in space	Postulate: The locus of points in space at a given distance from a given point is a sphere whose center is the given point and whose radius is the given distance. Postulate: The locus of points in space equidistant from two given points is the plane which is the perpendicular bisector of the line segment joining the two given points. Postulate: The locus of points in space at a given distance from a fixed line is a cylindrical surface with the line as an axis and a radius equal to the given distance. Th. The locus of points in space at a given distance from a given plane is a pair of planes each parallel to the given plane and at the given distance from it. Th. The locus of points in space equidistant from two parallel planes is a plane parallel to each of the given planes and midway between them.

In an equilateral triangle, the altitudes, medians, perpendicular bisectors of the sides, and angle bisectors coincide.
In an equilateral triangle, the incenter circumcenter, orthocenter, and centroid are all the same point.

An interesting problem involving locus is "In any triangle, find which three of the four centers are collinear."
Have pupils draw a triangle and then construct the incenter, circumcenter, orthocenter, and centroid.
Let them discover which three of these four points are collinear. (The incenter is not collinear to the other three.)
The proof of this problem is very involved and will make a good project for a superior student.

A (296)

B (selected exercises on 31 - 32, 39, 45, 70, 81, 139, 147, 153)

D (283 - 285)

E (476 - 483)

F (376 - 381)

G (70 - 72)

Other examples of loci in space:

What is the locus of:

1. points equidistant from two parallel walls?
2. points equidistant from two intersecting walls and two feet from the floor?
3. points equidistant from two points on the floor and two points on the chalkboard?
4. points equidistant from the floor and one wall and equidistant from the ceiling and the floor?
5. points on the floor at a given distance of five feet from a point on the wall four feet above the floor?
6. points equidistant from parallel planes R and S and a distance d from a third plane T not parallel to R?
7. points equidistant from the faces of a dihedral angle?
8. the center of a marble, one inch in diameter, that is free to roll on a horizontal plane surface?
9. points equidistant from two fixed points and at a given distance from a third point?
10. points equidistant from three non-collinear points?

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
SPECIAL LOCI (optional)	The introduction of conic sections offers an excellent opportunity for pupil projects. Such projects may include models, research papers, or original proofs. This topic is optional but should be called to the attention of the better pupils.

REFERENCES

D (295 - 298)

E (567 - 569)

F (390 - 391)

Johnson, D. A.
Curves in Space.
(16-48)

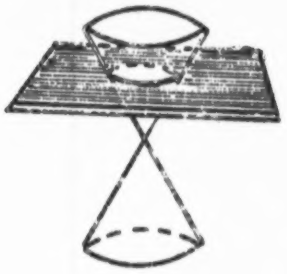
Lockwood, E. H.
The Book of Curves.
(2-33)

SUGGESTIONS

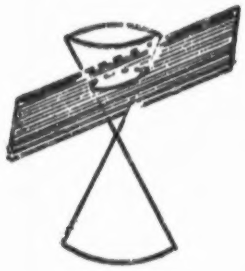
The Conic Sections - a good project for pupils.

Four different curves of intersection can be developed from the cutting of a conical surface, which has two nappes or branches, by a plane.

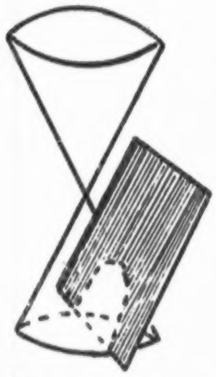
Cutting a cone parallel to the base, but not through the vertex, produces a circle.



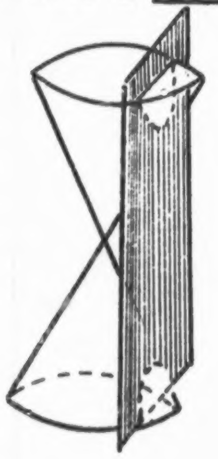
Cutting all the elements of a cone at an angle oblique to the base produces an ellipse.



Cutting a cone parallel to the slant height produces a parabola.



Cutting a cone parallel to the altitude and going through both nappes but not the vertex produces a hyperbola.

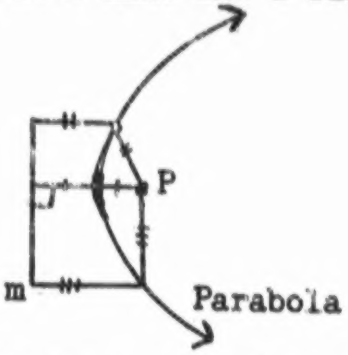


TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

REFERENCES	SUGGESTIONS

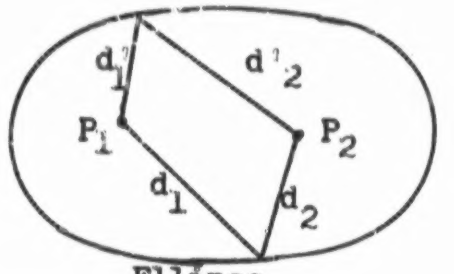
All four of these conics can be defined in terms of locus in a plane.
The circle has already been defined.

A parabola is the locus in a plane of points which are equidistant from a fixed line and a fixed point not in the line.



Parabola

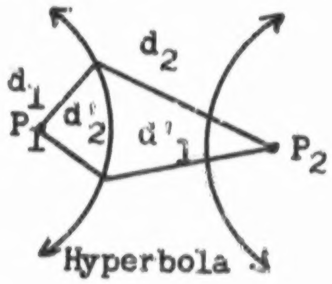
An ellipse is the locus of points in a plane the sum of whose distances from two fixed points is constant.



$d_1 + d_2 = c$

Ellipse

A hyperbola is the locus of points in a plane the difference of whose distances from two fixed points is constant.



$d_1 - d_2$ or

$d_2 - d_1 = c$

Hyperbola

In certain cases, the intersection of a cone and a plane is a point, a line, or a pair of intersecting lines. These intersections are known as degenerate conic sections.

These conics, taken either individually or collectively, make excellent material for projects.

UNIT XII

COORDINATE GEOMETRY

10 Sessions

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM	
INTRODUCTION TO COORDINATE GEOMETRY To acquaint pupils with the historical background of coordinate geometry		
COORDINATE SYSTEM To reinforce the concepts relating to points and numbers	**point in a line**	For every point in a line there exists one and only one real number. There is a one-to-one correspondence between the set of points in a line and the set of real numbers. The number which is used to label the point in the line is called the coordinate of the point in the line. This is a one-dimensional coordinate system.
	point in a plane	For every point in a plane there exists one and only one ordered pair of real numbers. There is a one-to-one correspondence between the set of points in a plane and the set of ordered pairs of real numbers. The coordinates of the point in the plane are the ordered pair of numbers used to label the point in the plane. This is a two-dimensional coordinate system. To assign an ordered pair of numbers to a point in a plane, use a pair of perpendicular number lines.
	x-axis	The x-axis is the horizontal number line.
	y-axis	The y-axis is the vertical number line.
	origin	The origin is the point of intersection of the x-axis and the y-axis. The coordinates of the origin are (0, 0).

REFERENCES	SUGGESTIONS
D (422 - 423) F (434) Bell, E. T. Men of Mathematics. (35-55)	Euclid, in his original presentation of geometry, designed it as preparation for philosophical study. He was not interested in the practical applications. It remained for Rene Descartes (1596-1650) to add immeasurably to the usefulness of geometry from a practical standpoint. Descartes was principally responsible for unifying algebra and geometry into the system which is known as Cartesian or coordinate geometry. His contribution was that of associating an ordered pair of numbers for every point in a plane. An oft-repeated story relates that Descartes developed coordinate geometry in an attempt to describe the path of a fly across the wall of his room. Actually, the essentials of Descartes' thinking had been used for many years before him for map making and navigational purposes.
A (485) C (294 - 299) D (423 - 426) F (393 - 404) Shute, W. C., Shirk, W. W., and Porter, G. F. Supplement to Plane Geometry. (3-9)	A review of the concepts pertinent to the study of coordinate geometry is necessary to the successful completion of this unit. The review should be brief but thorough. The past experience of the pupils will determine the extent of the review.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	x-coordinate The x-coordinate (abscissa) of a point is the number associated with the projection of the point on the x-axis.
	y-coordinate The y-coordinate (ordinate) of a point is the number associated with the projection of the point on the y-axis.
	quadrants The two axes separate the number plane into four regions called quadrants.
COORDINATE GEOMETRY METHODS To develop the understanding of the methods used in coordinate geometry	distance between two points 1. If the two points lie in the same vertical line, the distance between them is the absolute value of the difference between their y-coordinates. 2. If the two points lie in the same horizontal line, the distance between them is the absolute value of the difference between their x-coordinates. 3. A general method for finding the distance between any two points is derived by means of the Pythagorean Theorem: a. Find the difference between the x-coordinates and square this difference. b. Find the difference between the y-coordinates and square this difference. c. Add the two squares and compute the square root of their sum.

A (486 - 492)

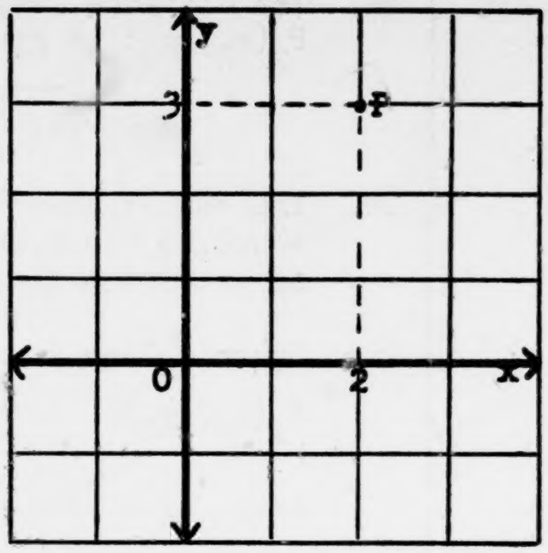
C (300 - 312)

D (426 - 438,
441 - 446)

G (405 - 433,
437 - 448)

Shute, W. G., Shirk, W. W., and Porter, G. F. Supplement to Plane Geometry. (10-16, 22-33)

Young, Frederick H. Pythagorean Numbers · Congruences, A Finite Arithmetic · Geometry in the Number Plane. (13-19)



The projection of point P on the x-axis is the point on the x-axis having an x-coordinate of 2.

The projection of point P on the y-axis is the point on the y-axis having a y-coordinate of 3.

The coordinates of point P are (2, 3).

Sample: Find the distance between (2, 3) and (7, 5).

Solution: a. $(7 - 2) = 5$

$5^2 = 25$

b. $(5 - 3) = 2$

$2^2 = 4$

c. $25 + 4 = 29$

$\sqrt{29} = 5.385$

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

Th. The distance between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Th. The coordinates of the midpoint of a line segment are equal to the average of the corresponding coordinates of the end points of the line segment.

Given a line segment whose end points are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $P_m(x_m, y_m)$ be the midpoint of the segment,

then $x_m = \dfrac{x_1 + x_2}{2}$ and $y_m = \dfrac{y_1 + y_2}{2}$

slope of a line

The slope of a line segment whose end points are (x_1, y_1) and (x_2, y_2) is determined by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a line segment is a number, either positive, negative, or zero.
The slope of a line is the same as the slope of any segment in the line.

If a line rises as the eye travels from left to right, the slope will be positive.
If the line falls, the slope will be negative.
If the line is horizontal, the slope is zero.
If the line is vertical, the slope is infinite.

If y increases as x increases, the slope is positive.
If y decreases as x increases, the slope is negative.

REFERENCES	SUGGESTIONS
	Note that in the distance formula, it does not matter which point is labeled P_1 and which is labeled P_2 since the difference is squared and the result is the same.
	The teacher may wish to introduce the concept of "rise" and "run". "Rise" is defined as the difference between the y-coordinates or the vertical change. "Run" is defined as the difference between the x-coordinates or the horizontal change. The "delta" notation may be introduced at this time. Δy is the "rise" or the change in y. Δx is the "run" or the change in x. The formula for the slope becomes: $m = \frac{\Delta y}{\Delta x}$

Th. Two non-vertical lines are parallel if they have the same slope and conversely.

Th. Two non-vertical lines are perpendicular if their slopes are negative reciprocals and conversely.

$$m_1 = \frac{-1}{m_2} \qquad \text{or} \qquad m_1 m_2 = -1$$

A locus is the set of points and only those points which satisfy one or more given conditions.
If these conditions are algebraic in nature, the locus is called a graph.
A graph is a picture of the solution set of an equation.

determining the equation of a line

Given: A point $P_1(x_1, y_1)$ and slope m of the line passing through P_1

To find: The equation of the line passing through P_1

Method: Take any point $P_n(x_n, y_n)$ in the given line.
Substitute the given values in the following formula:

$$y_n - y_1 = m(x_n - x_1)$$

(This formula is derived from the formula for finding the slope of a line.)

Given: Any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

To find: The equation of the line passing through P_1 and P_2

Method: Find slope m.
Substitute the given values in the formula:

$$y_n - y_1 = m(x_n - x_1)$$

---

x-intercept The x-intercept of a line is the x-coordinate of the line at the point where the line crosses the x-axis.

Sample problems in coordinate geometry involving equations of lines, parallel lines, perpendicular lines, and x and y-intercepts:

1. Find the equation of a line passing through (3, -4) and having a slope of -2.

   Solution:
   $$\begin{aligned} y - (-4) &= -2(x - 3) \\ y + 4 &= -2x + 6 \\ 2x + y &= 2 \end{aligned}$$

2. Find the equation of a line passing through (-1, -2) and (3, 8).

   Solution:
   $$\begin{aligned} m &= \frac{8 - (-2)}{3 - (-1)} \\ m &= \frac{10}{4} \\ m &= \frac{5}{2} \\ y - 8 &= \frac{5}{2}(x - 3) \\ 2y - 16 &= 5x - 15 \\ -5x + 2y &= 1 \\ 5x - 2y &= -1 \end{aligned}$$

3. Find the slope and the y-intercept of the line whose equation is $4x - y = -5$.

4. Find the equation of a line which passes through (-2, 3) and is parallel to the line whose equation is $2x = y + 7$.

y-intercept — The y-intercept of a line is the y-coordinate of the line at the point where the line crosses the y-axis.

determining the equation of a circle

Given: A circle with a center $P_1(x_1, y_1)$ and a radius r

To find: The equation of the circle with the given center and radius

Method: Take any point $P_n(x_n, y_n)$ in the circle.

By means of the distance formula determine the length of the line segment from P_1 to P_n.

Set this distance equal to the radius.

$$r = \sqrt{(x_n - x_1)^2 + (y_n - y_1)^2}$$

COORDINATE GEOMETRY PROOFS

To develop the ability to use coordinate geometry as a means of proving theorems

Th. (prove using coordinate geometry) The line connecting the midpoints of two sides of a triangle is parallel to the third side and equal to one-half its length.

A (493 - 494)

C (312 - 315)

D (438 - 441)

G (448 - 457)

Shute, W. C., Shirk, W. W., and Porter, G. F. Supplement to Plane Geometry. (17-18)

5. **Find the equation of a line which passes through the origin and is perpendicular to $ax + by = f$.**

Sample problem in coordinate geometry concerning the equation of a circle.

Find the equation of a circle whose center is (3, 4) and whose radius is 8.

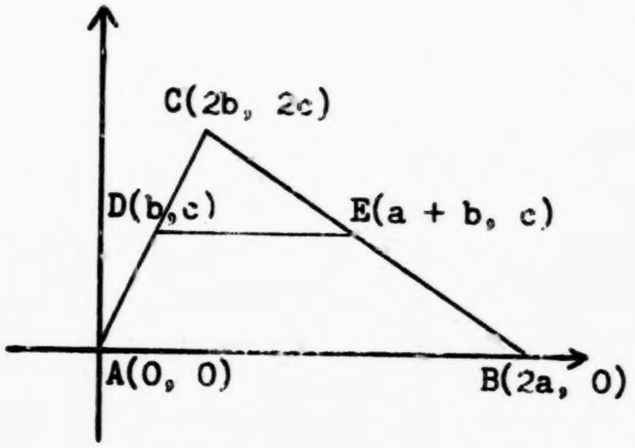
Solution:
$$8 = \sqrt{(x - 3)^2 + (y - 4)^2}$$
$$8 = \sqrt{x^2 - 6x + 9 + y^2 - 8y + 16}$$
$$64 = x^2 - 6x + 9 + y^2 - 8y + 16$$
$$39 = x^2 - 6x + y^2 - 8y$$

Some of the theorems of geometry can be proved more easily by coordinate geometry than by Euclidean geometry.

Given: $\triangle ABC$ with vertices A(0, 0), B(2a, 0) and C(2b, 2c)
(Note that the use of 2a, 2b, and 2c makes the algebra easier.)
D and E are the midpoints.

To prove: $DE \parallel AB$, $DE = \frac{1}{2}(AB)$

Plan: Determine the coordinates of the midpoints.
Show that DE and AB have the same slope.
Determine the length of DE and AB using the distance formula.

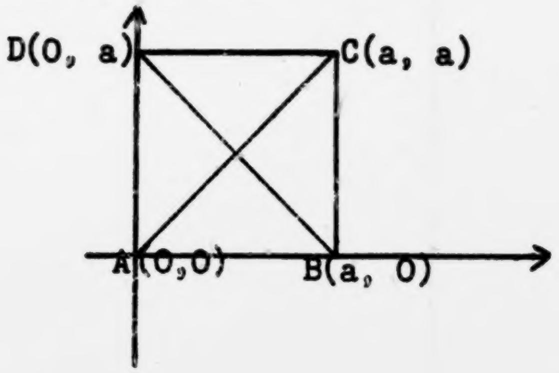


TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. (prove using coordinate geometry) The diagonals of a square are perpendicular to each other.
	Th. (prove using coordinate geometry) If line segments are drawn joining the midpoints of the sides of any quadrilateral, taken in order, the figure formed is a parallelogram.

Given: Square ABCD with vertices A(0, 0), B(a, 0), C(a, a), and D(0, a)

To prove: AC = BD

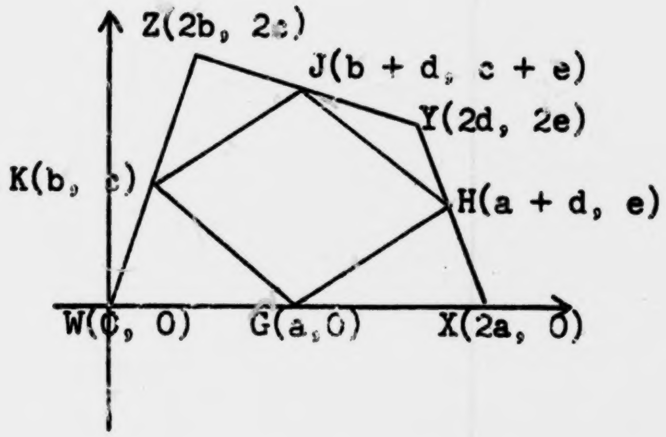
Plan: Show that the slope of AC is the negative reciprocal of the slope of BD.


D(0, a)
C(a, a)
A(0,0)
B(a, 0)

Given: Quadrilateral WXYZ with vertices W(0, 0), X(2a, 0), Y(2d, 2e), and Z(2b, 2c)
Midpoints are G, H, J, and K

To prove: Quadrilateral GHJK is a parallelogram

Plan: Find the coordinates of the midpoints.
Show GH and JK have the same slope.
Show KG and JH have the same slope.


Z(2b, 2c)
J(b + d, c + e)
Y(2d, 2e)
K(b, c)
H(a + d, e)
W(0, 0)
G(a, 0)
X(2a, 0)

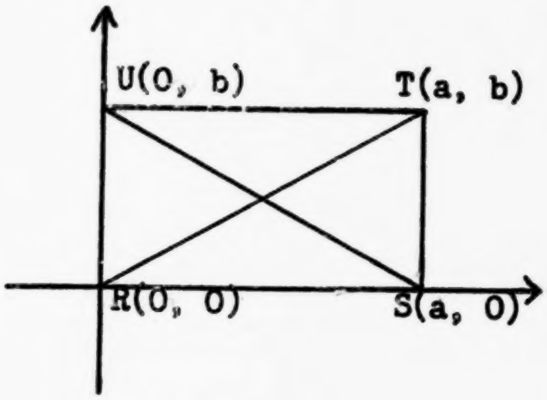
TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. (prove using coordinate geometry) The diagonals of a rectangle are equal.

REFERENCES	SUGGESTIONS

Given: Rectangle RSTU with vertices R(0, 0), S(a, 0), T(a, b), and U(0, b)

To prove: RT = SU

Plan: Determine the lengths of RT and SU by the distance formula.



Additional exercises that may be proved by methods of coordinate geometry:

1. The diagonals of a parallelogram bisect each other.
2. Two lines in the same plane perpendicular to the same line are parallel.
3. The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices of the triangle.
4. If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.
5. The segment joining the midpoints of the diagonals of a trapezoid is parallel to the bases.
6. If a line parallel to the bases of a trapezoid bisects one leg, it bisects the other leg also.
7. The perpendicular bisectors of the sides of any triangle meet in a point.

UNIT XIII

AREAS OF POLYGONS AND CIRCLES

9 Sessions

TOPICS AND OBJECTIVES

CONCEPTS, VOCABULARY, SYMBOLISM

AREAS OF POLYGONS

To develop an understanding of methods for finding the areas of certain polygons

unit of area

The unit of area is the surface of a plane enclosed by a square whose side is a unit of length.

If the unit of length is a foot, the unit of area is a square foot.

measure of a surface

The numerical measure of a surface is the number of times a unit of area is contained in the surface.

area of a polygon

The area of a polygon is the number of units of area contained in the surface bounded by the polygon.

equal polygons

Equal polygons are polygons that are equal in area.

Congruent polygons are both equal in area and similar.
All congruent polygons are equal. The converse is not true.

Postulate: The area of a rectangle is equal to the product of its base and its altitude when both are expressed in the same linear units.

$$A = bh$$

Th. The area of a parallelogram is equal to the product of its base and its altitude when both are expressed in the same linear units.

Corol. Parallelograms with equal bases and equal altitudes are equal in area.

Th. The area of a square is equal to the square of one of its sides.

$$A = s^2$$

Th. The area of a triangle is equal to one-half the product of its base and its altitude when both are expressed in the same linear units.

$$A = \frac{1}{2}(bh)$$

Corol. Triangles with equal bases and equal altitudes are equal in area.

REFERENCES

A (410 - 412)

C (259)

D (361 - 362)

E (417)

F (471 - 474)

A (415 - 426, 432 - 433)

C (261 - 275)

D (362 - 369)

E (417 - 438, Insert between pp. 312 and 313)

F (474 - 486, Insert between pp. 272 and 273)

SUGGESTIONS

Note: The symbol for congruent, " ≅ ", is an equal sign with the sign for "similar" above it.

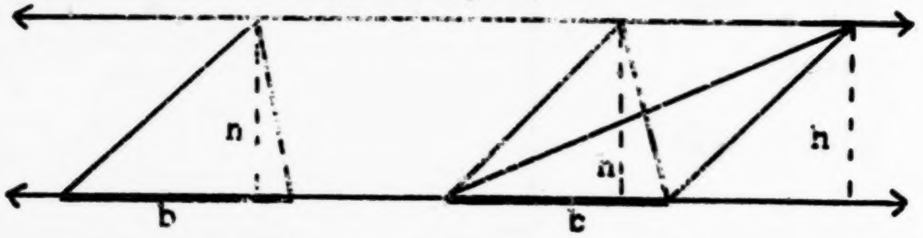
Pupils often enjoy developing formulas on their own. A good problem to assign pupils is the development of a formula for the area of an equilateral triangle whose side is s.
The formula is: $A = \frac{s^2\sqrt{3}}{4}$.

The formula can be developed through use of the general formula for the area of a triangle and the pupils' knowledge of the 30-60-90 degree triangle relationships.

TOPICS AND OBJECTIVES

CONCEPTS, VOCABULARY, SYMBOLISM

Corol. Triangles with a common base or equal bases in the same straight line and whose opposite vertices lie in a line parallel to the base, are equal.



Th. The area of a trapezoid is equal to one-half the product of its altitude and the sum of its bases.

$$A = \frac{1}{2}h(b + b')$$

Th. The area of a rhombus is equal to one-half the product of its diagonals.

$$A = \frac{1}{2}(D\ d)$$

Th. The area, K, of any triangle whose sides are a, b, and c is determined by the following formula:

$$K = \sqrt{s(s - a)(s - b)(s - c)},$$
$$\text{where } s = \frac{1}{2}(a + b + c)$$

Postulate: The areas of two similar polygons have the same ratio as the squares of any two corresponding linear dimensions.

Corresponding linear dimensions refer to corresponding sides, corresponding altitudes, corresponding diagonals, corresponding medians and others.

REGULAR POLYGONS AND THE CIRCLE

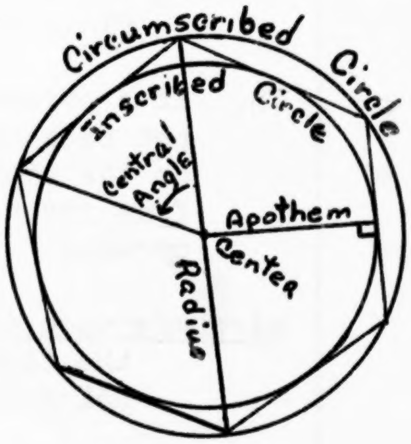
To develop an understanding of certain properties of regular polygons and circles

Th. A circle can be circumscribed about any regular polygon.

Th. A circle can be inscribed in any regular polygon.

center of a regular polygon The center of a regular polygon is the common center of its inscribed and circumscribed circles.

radius of a regular polygon The radius of a regular polygon is the radius of its circumscribed circle.

REFERENCES	SUGGESTIONS
	A geometric proof of the Pythagorean Theorem by areas is attributed to Euclid. The proof of this theorem by areas would be a good enrichment exercise.
	This formula is attributed to Hero of Alexandria who lived in the first century A.D. It is known as Hero's Formula. Hero is also known as Heron and the formula as Heron's Formula. The proof of this theorem is a good project for pupils.
A (437 - 455) C (415 - 420) D (370 - 371, 375 - 388) E (541 - 556) F (487 - 497)	 Circumscribed Circle Inscribed Circle Central Angle Apothem Center Radius

TOPICS AND OBJECTIVES | **CONCEPTS, VOCABULARY, SYMBOLISM**

__apothem of a regular polygon__ The apothem of a regular polygon is the radius of its inscribed circle drawn to the point of contact.

As the number of sides of the inscribed polygon increases, the length of the apothem increases and approaches the radius of the circumscribed circle as a limit.

__central angle of a regular polygon__ The central angle of a regular polygon is the angle at the center of the polygon whose sides are radii drawn to successive vertices.

Corol. The central angle of a regular polygon of n sides is equal to 360° ÷ n.

Corol. The apothem of a regular polygon is the perpendicular bisector of one side of the polygon.

Corol. The radius of a regular polygon bisects the angle to whose vertex it is drawn.

Th. Regular polygons of the same number of sides are similar.

Th. The perimeters of two regular polygons of the same number of sides have the same ratio as their radii or as their apothems.

Th. The area of a regular polygon is half the product of its apothem and its perimeter.

$$A = \frac{1}{2}ap$$

Corol. The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.

Th. If a pyramid is cut by a plane parallel to the base and not passing through a vertex, the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.

__circumference of a circle__ The circumference of a circle is the limit of the perimeters of the inscribed regular polygons.

As the number of sides of the inscribed polygon increases, the perimeter of the inscribed polygon increases and approaches the circumference of the circumscribed circle as a limit.

REFERENCES	SUGGESTIONS
	apothem (ap' o • them) is pronounced with the accent on the first syllable.
	The theorem regarding the area of a regular polygon may be proved as an exercise.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. The ratio of the circumference (C) of a circle to its diameter (d) is a constant, π. **Corol.** The circumference of a circle equals π times the diameter or 2π times the radius. **Corol.** The circumferences of two circles have the same ratio as their radii or their diameters.
PI To extend the concept of pi	

REFERENCES

SUGGESTIONS

π is an irrational number. Therefore, C and d cannot both be rational numbers.

A (453 - 454)

C (421 - 423)

D (385 - 386)

E (551, 570 - 571)

F (558 - 563, 496)

Davis, Philip J. The Lore of Large Numbers. (55-65)

Gamow, G. One, Two, Three, Infinity! (213-218)

Kasner, E., and Newman, James. Mathematics and the Imagination. (65-80)

Newman, James R. The World of Mathematics, Vol. I. (138)

Young, Frederick H. Random Numbers • Mathematical Induction • Geometric Numbers. (5-11)

THE TRANSCENDING NATURE OF π

A discussion of pi may be found in most geometry textbooks. This discussion generally includes a definition of pi as the limit of the ratio of the perimeter of an inscribed polygon to the radius of its circumscribed circle.

Often the text will mention the irrational nature of pi. There is generally some historical background on the development of an evaluation of pi. In various ways and by constantly improved methods, pi has been calculated to more and more decimal places.

The most recently known large scale calculation of pi was done in July, 1961 in New York. On an I.B.M. 7090 electronic computer, pi was computed to 100,000 decimal places in only eight hours.
Textbooks give the approximate value of pi from as few as 10 to as many as 5,000 decimal places. Here, generally, is where the discussion stops. Pupils tend to think of pi as a rather curious number related to circles and as a toy to be fed into electronic computers. But pi appears elsewhere in mathematics. An investigation of pi will give pupils an insight into the interrelatedness of all branches of mathematics.

Pi as the sum of an infinite series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

$$\frac{\pi}{4} = \frac{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \ldots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \ldots}$$

$$\frac{4}{\pi} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ldots}}}}}$$

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

REFERENCES	SUGGESTIONS

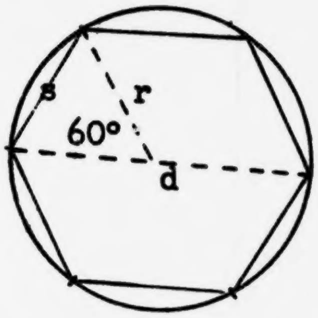
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\pi = 2^n\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}$$ as n increases without bound

In this series, n is the number of sides of an inscribed regular polygon. The series is derived from approximations of π determined by the ratio of the perimeter (p) of each n-gon to the diameter of the circumscribed circle.

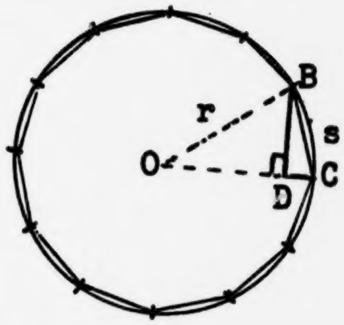
A 6-gon inscribed in a circle whose diameter is 4.

$d = 4$
$r = 2$
$s = 2$
$P = 6s$
$P = 12$
$\frac{P}{d} = \frac{12}{4}$
$\therefore \frac{P}{d} = 3$



A 12-gon inscribed in a circle whose diameter is 4.

$d = 4$
$r = 2$
$\angle BOC = 30°$
$OB = 2$
$BD = 1$
$OD = \sqrt{3}$
$DC = 2 - \sqrt{3}$
$BC = \sqrt{1 + 4 - 4\sqrt{3} + 3}$
$BC = 2\sqrt{2 - \sqrt{3}}$
$BC = s$
$s = 2\sqrt{2 - \sqrt{3}}$
$P = 12s$
$P = 24\sqrt{2 - \sqrt{3}}$
$\frac{P}{d} = \frac{24\sqrt{2 - \sqrt{3}}}{4}$
$\therefore \frac{P}{d} = 3.106$

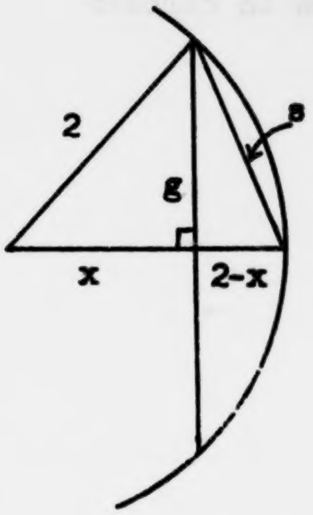


TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

A general formula for finding the side of a regular 2n -gon inscribed in a circle having a diameter of 4 units when the side of a regular n-gon is known

Let $g = \frac{1}{2}$ the side of a regular n-gon

Let s = the side of a regular 2n-gon



$$s^2 = g^2 + (2 - x)^2$$
$$s^2 = g^2 + 4 - 4x + x^2$$
$$\text{but } x^2 = 4 - g^2$$
$$x = \sqrt{4 - g^2}$$
$$\text{then } s^2 = g^2 + 4 - 4\sqrt{4 - g^2} + 4 - g^2$$
$$s^2 = 4(2 - \sqrt{4 - g^2})$$
$$\therefore s = 2\sqrt{2 - \sqrt{4 - g^2}}$$

Pi as a measure of statistical probability:

Count Buffon's Needle Experiment:

1. Distribute some flat toothpicks, all of the same length, one to each pupil.
2. Provide each pupil with a sheet of blank paper.
3. Have each pupil draw equidistant parallel lines so that the entire surface is ruled. The lines should be the same distance apart as the length of the toothpick.
4. Instruct each pupil to drop the toothpick on the paper 100 times. (The pupils can do this at home.)
5. Have each pupil tabulate the number of times the toothpick, when dropped on the paper, lies on or across one of the rulings.

   If any part of the toothpick lies across a ruling, it is to be counted as a success. If the toothpick lies entirely between rulings, it is to be counted as a failure.

   Occasions when the toothpick does not fall on the paper are not to be counted.

6. Tabulate the total number of "successes". The ratio of the number of successes to the number of trials will equal $\frac{2}{\pi}$.

A biologist investigating laws of bacterial growth and an insurance actuary computing probability both use pi in their work.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

AREA OF CIRCLES

To extend the concept of area to circles

area of a circle — The area of a circle is the limit of the areas of the inscribed regular polygons.

As the number of sides of the inscribed regular polygon increases, the area of the inscribed polygon increases and approaches the area of the circle as a limit.

Th. The area of a circle is equal to the product of π and the square of the radius.

Corol. The areas of two circles have the same ratio as the squares of their radii, the squares of their diameters, or the squares of their circumferences.

sector of a circle — A sector of a circle is a region bounded by two radii and their intercepted arc.

Corol. The area of a sector of a circle whose radius is r and whose intercepted arc contains n degrees is determined by the following formula:

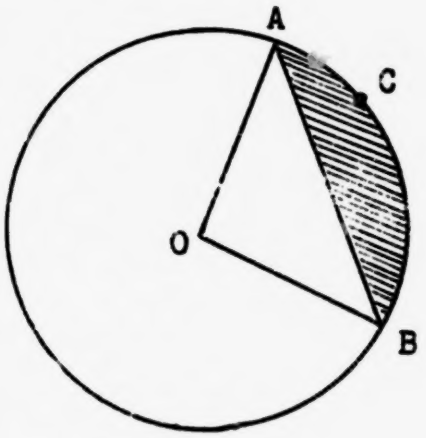
$$\text{Area of sector} = \frac{n}{360}\pi r^2$$

segment of a circle — A segment of a circle is a region bounded by a chord and its intercepted arc. The area of a segment can be found by subtracting the area of a triangle from the area of a sector.

REFERENCES	
A	(456 - 457, 459 - 462)
C	(421 - 430)
D	(388 - 391)
E	(556 - 559)
F	(498 - 503)

SUGGESTIONS

The area of segment ACB = area of sector AOBC minus area of triangle AOB.



UNIT XIV

GEOMETRIC SOLIDS - AREAS AND VOLUMES

10 Sessions

TOPICS AND OBJECTIVES | **CONCEPTS, VOCABULARY, SYMBOLISM**

PRISMS AND PYRAMIDS

To develop an understanding of area and volume of prisms

lateral area of a prism The lateral area of a prism is the sum of the areas of the lateral faces.

total area of a prism The total area of a prism is the sum of the lateral area and the areas of the bases.

Th. The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.

$L.A. = ep$, where e is the lateral edge and p is the perimeter of the right section.

Corol. The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.

unit of volume The unit of volume is the space enclosed by a cube whose side is a unit of length.

If the unit of length is a foot, the unit of volume is a cubic foot.

volume of a solid The volume of a solid is the number of units of volume contained in the space enclosed by the solid.

equal solids Equal solids are solids that are equal in volume.

Postulate: The volume of a rectangular solid equals the product of its three dimensions.

$V = abc$, where a, b, and c are the dimensions.

Th. The volume of any prism is equal to the product of the area of its base and its altitude.

To develop an understanding of area and volume of pyramids

lateral area of a pyramid The lateral area of a pyramid is the sum of the areas of the lateral faces.

total area of a pyramid The total area of a pyramid is the sum of the lateral area and the area of the base.

Th. The lateral area of a regular pyramid is equal to one-half the product of the slant height and the perimeter of its base.

$L.A. = \frac{1}{2}lp$, where l is the slant height and p is the perimeter of the base.

REFERENCES	SUGGESTIONS
B (91 - 96, 99 - 108 selected exercises) C (277 - 279) D (392 - 402) E (443 - 444, 563 - 564) F (519 - 521) G (34 - 40)	A brief review of the vocabulary pertinent to prisms and pyramids is advisable. A list of vocabulary and properties of pyramids and prisms will be found in Unit V. A formula which may be used for finding the total area of a prism is: $T.A. = ep + 2B$ Film: Surface Areas of Solids, Parts I and II (2 reels - total time - 36 min.) Cenco Educational Films Central Scientific Company 1700 Irving Park Road Chicago 13, Illinois Film: Volumes of Cubes, Prisms and Cylinders (Color - 15 min.) Colburn Film Distributors, Inc. P. O. Box 470 Lake Forest, Illinois Film: Volumes of Pyramids, Cones, and Spheres (Color - 15 min.) Delta Film Production 7238 West Touhy Avenue Chicago 48, Illinois
B (122 - 137 selected exercises) C (277) D (407 - 414) E (444 - 445, 565 - 566 selected exercises) F (521 - 523, inserts between pp. 528-529)	A formula which may be used for finding the total area of a pyramid is: $T.A. = \frac{1}{2}lp + B$

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	Th. The lateral area of a frustum of a regular pyramid is equal to one-half the product of the slant height and the sum of the perimeters of the bases. $\text{L.A.} = \frac{1}{2}l(p + p')$, where l is the slant height and p and p' are the perimeters of the bases.
	Th. If two solids are included between the same two parallel planes and if any plane parallel to these planes makes equal sections of the solids, the solids have equal volumes. (This theorem is known as Cavalieri's Theorem.)
	Postulate: Two pyramids having equal altitudes and equal bases are equal.
	Th. The volume of a triangular pyramid is equal to one-third of the product of its base and altitude.
	Corol. The volume of any pyramid is equal to one-third of the product of its base and altitude. $V = \frac{1}{3}Bh$, where B is the area of the base and h is the altitude of the pyramid.
	Th. The volume of the frustum of a pyramid is determined by the formula: $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$, where h is the altitude and B and B' are the areas of the bases.
	Th. The lateral areas or total areas of any two similar solids are in the same ratio as the squares of any two corresponding linear dimensions.
	Th. The volumes of any two similar solids are in the same ratio as the cubes of any two corresponding linear dimensions.
CYLINDERS AND CONES	
To develop an understanding of the vocabulary involving cylinders	circular cylinder — A circular cylinder is a cylinder whose bases are equal circles which lie in parallel planes.
	axis of a cylinder — The axis of a cylinder is the line segment whose end points are the centers of the bases.

REFERENCES	SUGGESTIONS
G (42 - 46)	
	Bonaventura Cavalieri (1598-1647) developed the proof of this theorem.
	The theorem concerning the volume of a triangular pyramid may be proved formally as an exercise using models if available.
	Since any pyramid can be divided into a whole number of triangular pyramids, the sum of the volumes of the triangular pyramids equals $\frac{1}{3}h(B_1 + B_2 + B_3 + \dots)$, where $B_1 + B_2 + B_3 + \dots$ equals B, the base of any pyramid.
	The two theorems concerning similar solids are syntheses of a number of theorems concerning particular types of solids. It is suggested that the teacher provide pupils with a sufficient number of exercises based on these theorems.
B (85 - 88 selected exercises) C (447 - 448) D (269 - 270, 404) E (359 - 360)	Cylinders and cones should be described informally rather than defined. The definition of cylinders and cones requires an understanding of cylindrical and conical surfaces. These, in turn, require an understanding of concepts such as generatrix, directrix, ruled surface, and nappes. Such concepts are not necessary to the successful completion of this unit.

altitude of a cylinder The altitude of a cylinder is the length of a common perpendicular between the planes of the bases.

right cylinder A right cylinder is a cylinder in which the planes of the bases are perpendicular to the axis.

oblique cylinder An oblique cylinder is a cylinder in which the planes of the bases are not perpendicular to the axis.

cylinder of revolution A right circular cylinder is called a cylinder of revolution because it may be formed by revolving a rectangle about one of its sides as an axis.

The axis and the altitude of a cylinder of revolution are equal.

To develop an understanding of area and volume of cylinders

lateral area of a cylinder The lateral area of a cylinder is the area of the curved surface.

total area of a cylinder The total area of a cylinder is the sum of the lateral area and the area of the bases.

Postulate: The lateral area of a cylinder is equal to the product of the axis and the perimeter of a right section.

$L.A. = ap$, where a is the axis and p is the perimeter of a right section of the cylinder.

Corol. The lateral area of a cylinder of revolution is equal to the product of the circumference of the base and the altitude.

$L.A. = Ca$ or $L.A. = 2\pi ra$

Th. The volume of a cylinder is equal to the product of the area of its base and the altitude.

$V = Bh$ or $V = \pi r^2 h$, where r is the radius of the base and h is the altitude.

REFERENCES	SUGGESTIONS
F (524 - 525) G (47 - 50)	A cylinder or cone may have any closed curve as a base. Since the circle is the only closed curve studied in detail in plane geometry, only circular cylinders and circular cones will be discussed. Use of the word cylinder implies circular cylinder unless otherwise stated. A circular cylinder may be thought of as a prism with a regular polygon as its base and an infinite number of lateral faces.
B (92 - 110 selected exercises) C (447 - 448) D (404 - 406) E (561 - 563, 565 - 566 selected exercises) F (525 - 526)	Have pupils make a comparison between the formulas for area and volume of cylinders and the formulas for the area and volume of prisms. They are essentially the same.
G (73 - 76)	A formula which may be used for finding the total area of a cylinder of revolution is: $T.A. = 2\pi ra + 2\pi r^2$ or $T.A. = 2\pi r(a + r)$

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
To develop an understanding of the vocabulary involving cones	circular cone A circular cone is a cone whose base is a circle. axis of a cone The axis of a cone is a line segment whose end points are the vertex of the cone and the center of the base. altitude of a cone The altitude of a cone is the perpendicular distance from the vertex to the plane of the base. right cone A right cone is a cone whose axis is perpendicular to the plane of the base. oblique cone An oblique cone is a cone whose axis is not perpendicular to the plane of the base. cone of revolution A right circular cone is called a cone of revolution because it may be formed by revolving a right triangle about one of its legs as an axis. The axis and the altitude of a cone of revolution are equal. slant height of a cone of revolution The slant height of a cone of revolution is a line segment whose end points are the vertex of the cone and any point in the circumference of the base.
To develop an understanding of the area and volume of cones	lateral area of a cone The lateral area of a cone is the area of the curved surface. total area of a cone The total area of a cone is the sum of the lateral area and the area of the base. Corol. The lateral area of a cone of revolution is equal to one-half the product of the slant height and the circumference of the base. L.A. = $\frac{1}{2}Cl$, where l is the slant height or L.A. = πrl, where r is the radius of the base.

REFERENCES	SUGGESTIONS
B (111 - 121 selected exercises) C (449 - 450) D (276 - 278) E (359 - 361) F (524) G (50 - 52)	Use of the word cone implies circular cone unless otherwise stated. A circular cone may be thought of as a pyramid with a regular polygon for a base and an infinite number of lateral faces.
B (121 - 143 selected exercises) C (449 - 451 selected exercises) D (414 - 419) E (561 - 562, 565 - 566 selected exercises) F (527 - 528, inserts between pp. 528-529) G (76 - 78)	Have pupils make a comparison between the formulas for area and volume of pyramids and the formulas for the area and volume of cones. They are essentially the same. A formula which may be used for finding the total area of a cone of revolution is: T.A. = $\pi rl + \pi r^2$ or T.A. = $\pi r(l + r)$

frustum of a cone: The frustum of a cone is the part of a cone included between the base and a plane parallel to the base.

Corol. The lateral area of the frustum of a cone of revolution is equal to one-half the product of the slant height and the sum of the circumferences of the bases.

$L.A. = \frac{1}{2}l(C + C')$, where l is the slant height and C and C' are the circumferences of the bases.

Corol. The volume of a cone is equal to one-third the product of the area of the base and the altitude.

$V = \frac{1}{3}Bh$ or $V = \frac{1}{3}\pi r^2 h$

Corol. The volume of a frustum of a cone is determined by the formula:

$V = \frac{1}{3}h(B + B' + \sqrt{BB'})$, where B and B' are the areas of the bases.

SPHERES

To develop an understanding of area and volume of spheres

Th. The area of a sphere is equal to the area of four great circles.

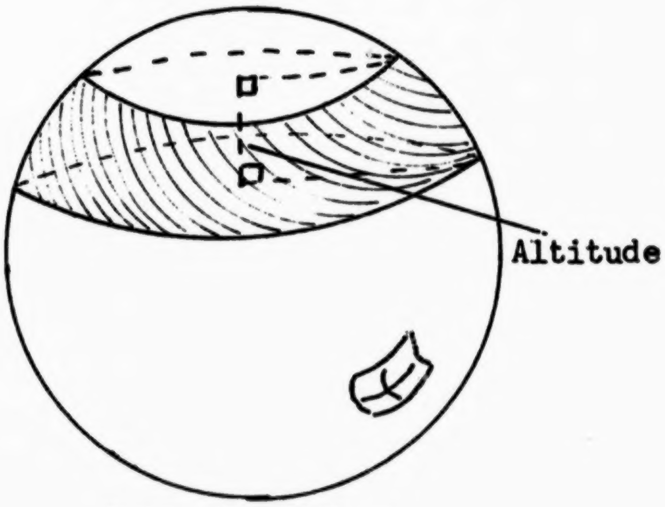
$S = 4\pi r^2$, where S is the area of the sphere and r is the radius of the great circle.

The radius of a great circle is the same as the radius of a sphere.

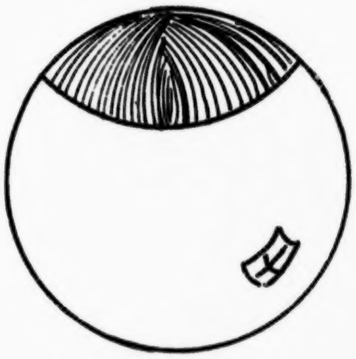
zone of a sphere: The zone of a sphere is the portion of the surface of the sphere included between two parallel planes.

The circles that bound a zone are called the bases of the zone.
The distance between the parallel planes is called the altitude of the zone.
If one of the parallel planes is tangent to the sphere, the zone is called a zone of one base or a dome.

REFERENCES	SUGGESTIONS
B (155 - 171) C (452 - 453) D (419 - 421) E (562, 565 - 566) F (528 - 530, 533 - 535) G (78 - 80)	A brief review of the vocabulary pertinent to spheres is advisable. A list of vocabulary and properties of spheres will be found in Unit IX.



Zone of a Sphere



Dome

TOPICS AND OBJECTIVES	CONCEPTS. VOCABULARY, SYMBOLISM
	Corol. The area of a zone is equal to the product of its altitude and the circumference of a great circle. $Z = 2\pi rh$, where r is the radius of the great circle and h is the altitude of the zone. **Th.** The volume of a sphere is determined by the following formula: $V = \frac{4}{3}\pi r^3$

REFERENCES	SUGGESTIONS

UNIT XV

SPHERICAL GEOMETRY

9 Sessions

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM

FIGURES ON A SPHERE

To develop an understanding of the nature of spherical distance

Postulate: The shortest distance between any two points in a sphere is the minor arc of the great circle through these points.

spherical distance The spherical distance between two points in a sphere is the length of the minor arc of a great circle joining the two points.

The measure of spherical distance may be in linear units or in degrees.

poles of a circle of a sphere The poles of a circle of a sphere are the intersections of the axis of the circle and the surface of the sphere.

Set A = all the points in the sphere
Set B = all the points in the axis of a circle
$A \cap B$ = two points called the poles of a circle of a sphere

polar distance of a circle of a sphere The polar distance of a circle of a sphere is the spherical distance from any point in the circle to the nearer pole.

quadrant A quadrant is an arc which is one-fourth of the circumference of a great circle. The arc is 90°.

Th. The spherical distances of all points in a circle of a sphere from either pole of the circle, are equal.

Corol. On the same or equal spheres, equal circles have equal polar distances.

Corol. The polar distance of a great circle is a quadrant.

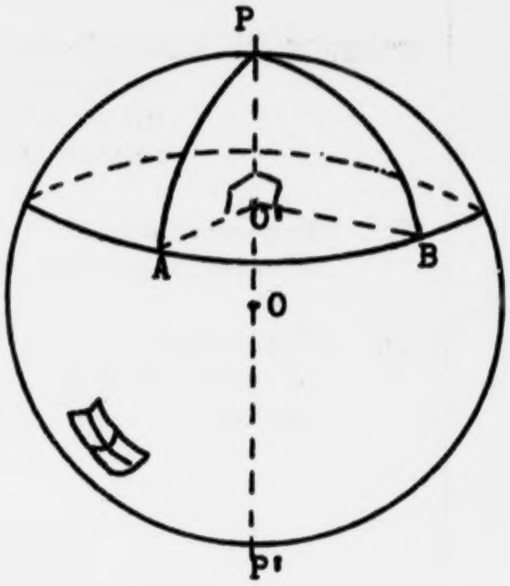
REFERENCES

B (172 - 173)

G (56 - 57)

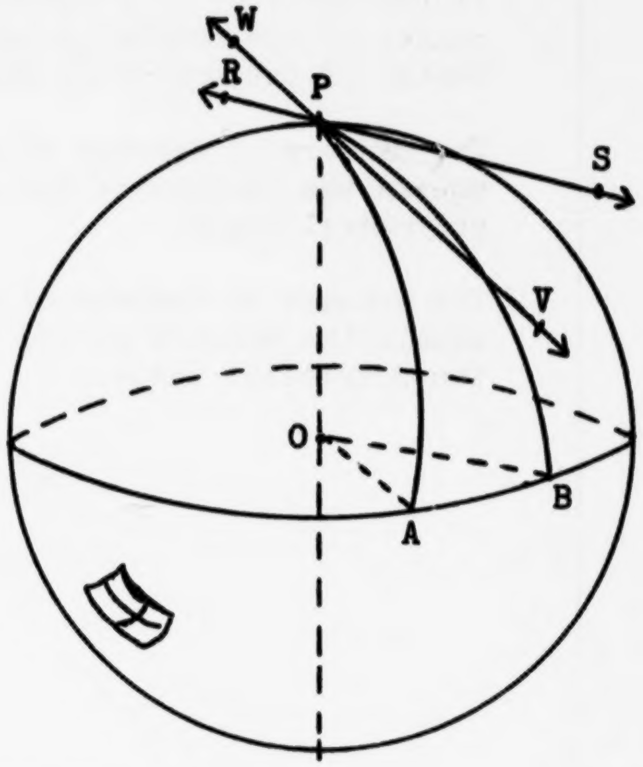
Henderson, K. B.,
Pingry, R. E.,
Robinson, G. A. Modern Geometry, Its Structure and Function. (452-456)

SUGGESTIONS



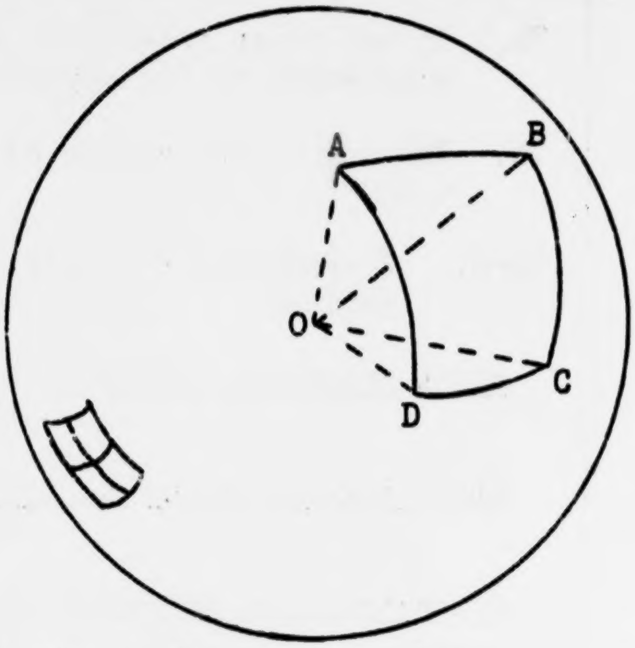
O' is a circle of sphere O.
Line PP' is the axis of circle O'.
P and P' are the poles of circle O'.
$\widehat{PA}$ and $\widehat{PB}$ are the polar distances of circle O'.
$\widehat{PA} = \widehat{PB}$

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
To develop an understanding of the nature of spherical angles	spherical angle A spherical angle is the union of two great circle arcs with a common end point. measure of a spherical angle The measure of a spherical angle is equal to the measure of the angle formed by the union of two lines tangent to the arcs at their common end point. Th. The measure of a spherical angle is equal to the measure of the dihedral angle formed by the planes of its sides. Th. The measure of a spherical angle is equal to the measure of the arc that it intercepts on a circle whose pole is the vertex of the spherical angle.
To develop an understanding of the nature of spherical polygons	spherical polygon A spherical polygon is a closed figure in a sphere formed by minor arcs of great circles. The arcs of the great circles are the sides of the polygon. The end points of the arcs are the vertices of the polygon. The diagonal of a spherical polygon is an arc of a great circle joining two nonconsecutive vertices. The angles of the polygon are the spherical angles formed by the sides of the polygon. All spherical polygons are to be considered as convex unless otherwise stated.

REFERENCES	SUGGESTIONS
B (174 - 175) G (60 - 61) Henderson, K. B., Pingry, R. E., Robinson, G. A. Modern Geometry, Its Structure and Function. (456-459) Schacht, J. F., McLennan, R. C., and Griswold, A. L. Contemporary Geometry. (481)	 $\angle APB$ is a spherical angle. WV is tangent to the plane of arc AP at P. RS is tangent to the plane of arc BP at P. The measure of spherical angle APB is defined as the measure of $\angle VPS$. The measure of spherical angle APB is equal in degrees to the measure of dihedral angle A-OP-B. The measure of spherical angle APB is equal in degrees to the measure of $\overset{\frown}{AB}$. Note: Spherical angles are adjacent, vertical, supplementary, complementary, obtuse, or acute under the same conditions as plane angles. Two great circle arcs are perpendicular if they form a spherical right angle.
B (175 - 180) G (65 - 66, 91 - 93)	A review of the properties and theorems pertaining to polyhedral angles should be integrated with the study of spherical polygons. The review material will be found in Units V and VI. Pupils should be able to discover the similarities between polyhedral angles and spherical polygons. These similarities will aid in the understanding of spherical polygons.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	If the vertices of a spherical polygon are joined to the center of the sphere, a polyhedral angle whose vertex is the center of the sphere is formed. The measure in degrees of a side of a spherical polygon equals the measure of the corresponding face angle of the polyhedral angle. The measure in degrees of an angle of a spherical polygon equals the measure of the corresponding dihedral angle of the polyhedral angle.
	spherical triangle A spherical triangle is a spherical polygon of three sides. Th. The sum of two sides of a spherical triangle is greater than the third side. Th. The sum of the sides of a spherical polygon $< 360°$.
To develop an understanding of the nature of polar triangles	polar triangle If the vertices of one spherical triangle are the poles of the sides of another spherical triangle, the second triangle is the polar triangle of the first triangle. The spherical distance from any vertex in a spherical triangle to the side opposite in the polar triangle is a quadrant.
	Th. If one spherical triangle is the polar triangle of another, then the second triangle is the polar triangle of the first. complementary and supplementary angles and arcs A spherical angle and an arc of a great circle are complementary or supplementary if their sum is 90° or 180°, respectively.

REFERENCES	SUGGESTIONS



O is the center of the sphere.
ABCD is a spherical polygon.
The measure of side AD is equal in degrees to the measure of $\angle AOD$.
The measure of spherical angle ADC is equal in degrees to the measure of dihedral angle A-OD-C.

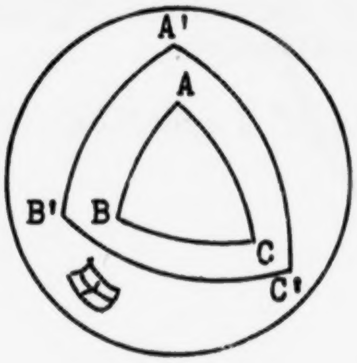
Note: Spherical triangles are isosceles, equilateral, equiangular, right, or obtuse in the same sense as triangles in planegeometry. The words median, altitude, and bisector of an angle have the same relative meaning. All spherical polygons have sides which are minor arcs of great circles.

B (180 - 184)

G (67 - 69, 92 - 93)

Pupils often have difficulty in grasping the concept of polar triangles. An alternate and perhaps easier method of locating the polar triangle of a spherical triangle is as follows:

1. Begin with the sides of the spherical triangle.
2. Locate the poles of the great circles of which the three sides are arcs.
3. Connect the poles with arcs of great circles.
4. A polar triangle is formed.



△ ABC and △ A'B'C' are spherical triangles. Each is the polar triangle of the other.

Th. In two polar triangles, each angle of one triangle is the supplement of the opposite side of the other.

Th. The sum of the angles of a spherical triangle $> 180^\circ$ but $< 540^\circ$.

Corol. A spherical triangle may have one, two, or three right angles.

right spherical triangle   A right spherical triangle is one that has at least one right angle.

birectangular spherical triangle   A birectangular spherical triangle has two right angles.

trirectangular spherical triangle   A trirectangular spherical triangle has three right angles.

Corol. A spherical triangle may have one, two, or three obtuse angles.

AREA OF SPHERICAL POLYGONS

To develop an understanding of the method of measuring the area of spherical polygons

spherical degree   A spherical degree is the area of a birectangular spherical triangle whose third angle is one degree.

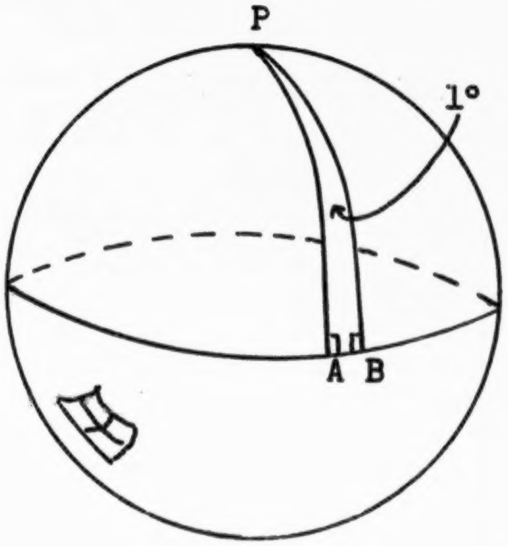
spherical excess of a spherical triangle   The spherical excess of a spherical triangle is the difference between the sum of the angles of the triangle and 180°.

spherical excess of a spherical polygon   The spherical excess of a spherical polygon is the difference between the sum of the angles of the spherical polygon and the sum of the angles of a plane polygon with the same number of sides.

Th. The area of a sphere equals 720 spherical degrees.

Th. The area of a spherical triangle in spherical degrees equals its spherical excess.

Corol. The area of a spherical polygon equals its spherical excess.

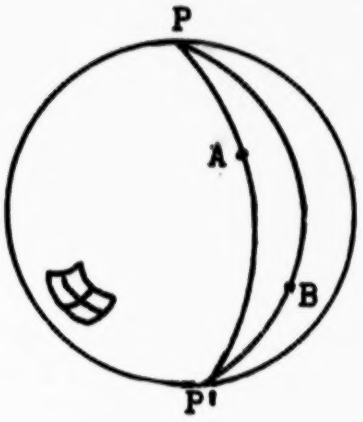
REFERENCES	SUGGESTIONS
	This theorem may be proved as an exercise. This theorem may be proved as an exercise.
B (193 - 204, selected exercises) G (93 - 95) Schacht, J. F., McLennan, R. C., and Griswold, A. L. Contemporary Geometry. (482-486)	 The area of spherical triangle PAB is equal to one spherical degree.

TOPICS AND OBJECTIVES	CONCEPTS, VOCABULARY, SYMBOLISM
	lune A lune is the surface of a sphere bounded by two great semicircles. Th. The number of spherical degrees in a lune equals twice the number of degrees in its angle.
(OPTIONAL) VOLUMES OF SPHERICAL SOLIDS	
To develop an understanding of the nature of certain spherical solids	spherical solid A spherical solid is a solid whose base is a portion of the surface of a sphere. The solid is formed by connecting every point in the perimeter of the base to the center of the sphere. spherical pyramid A spherical pyramid is a spherical solid whose base is a spherical polygon. spherical cone A spherical cone is a spherical solid whose base is a dome. spherical sector A spherical sector is a spherical solid whose base is a zone. spherical wedge A spherical wedge is a spherical solid whose base is a lune.
To develop an understanding of the method of determining the volume of certain spherical solids	Th. The volume of any spherical solid is equal to one third the product of the area of the base measured in square units and the radius of the sphere. $V = \frac{1}{3}Sr$, where S is the area of the base and r is the radius of the sphere.

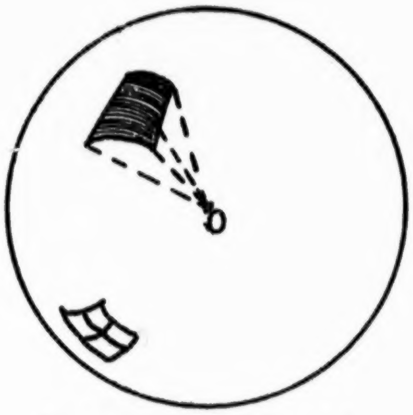
REFERENCES

SUGGESTIONS

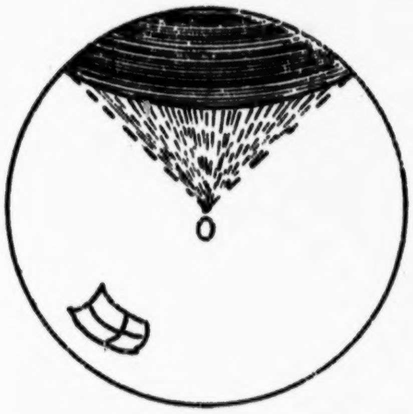
B (162 - 163)



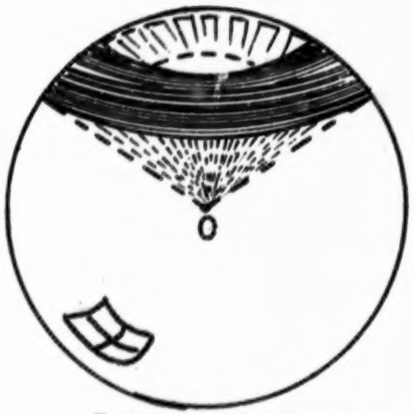
PAP'B is a lune.



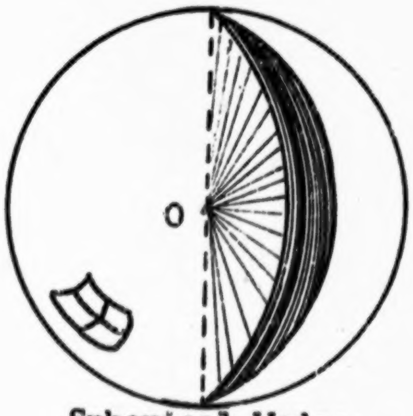
Spherical Pyramid



Spherical Cone



Spherical Sector



Spherical Wedge

B (163)

Note: The volume of a sphere can be derived directly from this theorem. If the base of the spherical solid is the entire surface of the sphere, then $S = 4\pi r^2$.

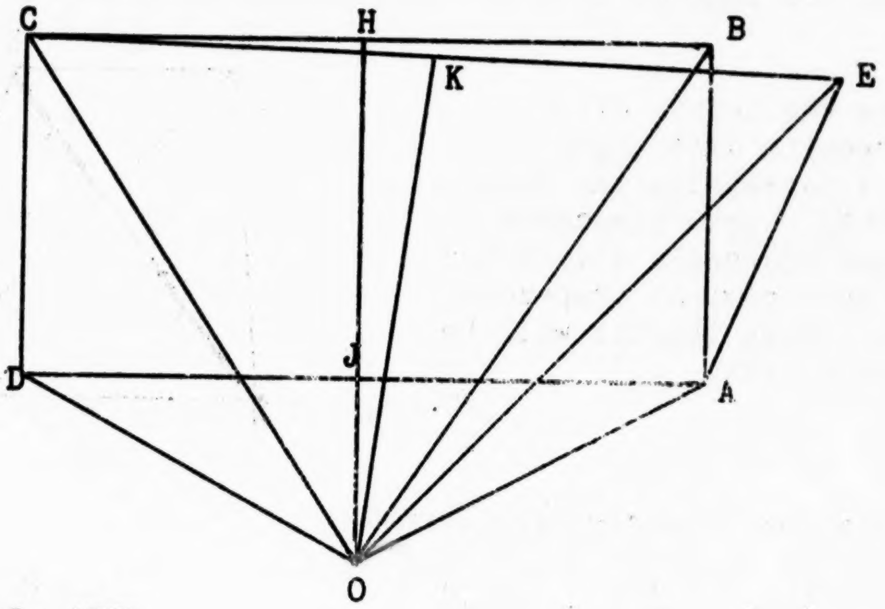
$V = \frac{1}{3}(4\pi r^2)(r)$ or,

$V = \frac{4}{3}\pi r^3$

APPENDIX

FIND THE FALLACY

The following is a fallacious proof--namely, a right angle is equal to an angle greater than a right angle!



Given: Rectangle ABCD
$AE = AB$
HJ $\perp$ bisector of CB and DA
K $\perp$ bisector of CE

To prove: $\angle JDC = \angle JAE$

Plan: H is the midpoint of BC. At H, draw a line $\perp$ to CB. This line will also be the $\perp$ bisector of AD and intersect AD at J. Now from A draw line AE outside the rectangle but equal in length to AB and CD. Draw line CE. At K, midpoint of CE, draw a line $\perp$ to CE. Extend HJ and the $\perp$ line through K to meet at O. Draw OA, OD, OB, $\overline{OC}$, and OE.

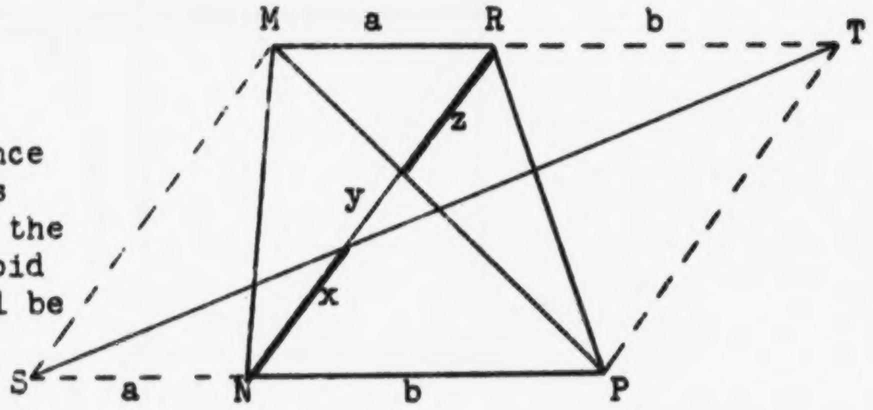
Statements	Reasons
1. $OE = OC$	1. OK is $\perp$ bisector of CE. Lines drawn from a point on the $\perp$ bisector of a line to the extremities of the line are equal.
2. $OA = OD$	2. Same reason as No. 1
3. $AE = CD$	3. So drawn
4. $\triangle ODC \cong \triangle OAE$	4. s.s.s. = s.s.s.
5. $\angle ODC = \angle OAE$	5. C.p.c.t.e.
6. $\angle ODA = \angle OAD$	6. $OA = OD$, and base angles at the foot of an isosceles triangle are equal.
7. $\therefore \angle JDC = \angle JAE$	7. Subtraction axiom

BUT, $\angle JDC$ is a right angle, and $\angle JAE$ is greater than a right angle!

FIND THE FALLACY

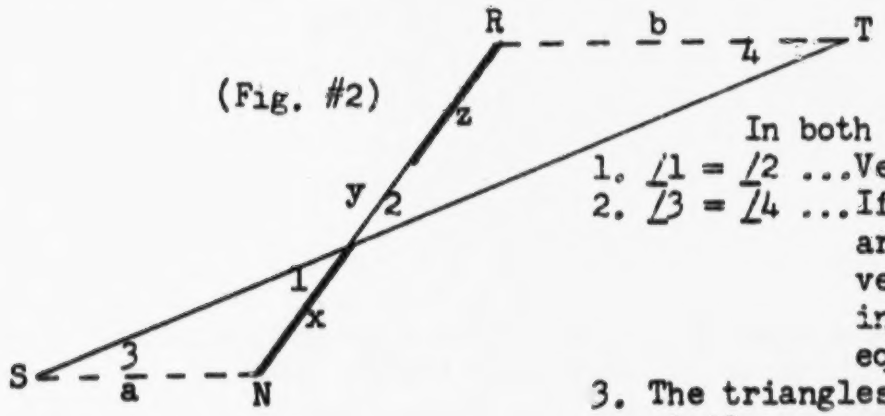
Proposition: The sum of the bases of a trapezoid is equal to zero!

Given: Trapezoid MNPR, the lengths of whose bases shall be called a and b.
To prove: $a + b = 0$



Plan: Extend each base the length of the other and in opposite directions. Figure MSPT is a parallelogram, since SP = MT and SP ∥ MT. Draw diagonals ST and MP. These diagonals divide the diagonal RN of the original trapezoid into three parts whose lengths will be called x, y, and z. (Fig. #1)

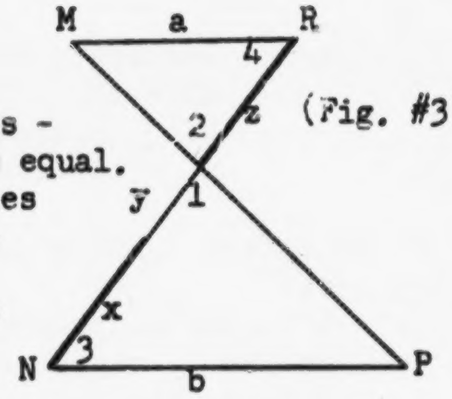
Proof. Two pairs of similar triangles are formed:



(Fig. #2)

In both pairs of triangles -

1. ∠1 = ∠2 ...Vertical angles are equal.
2. ∠3 = ∠4 ...If two parallel lines are cut by a transversal, alternate interior angles are equal.
3. The triangles are similar ... a.a. = a.a.



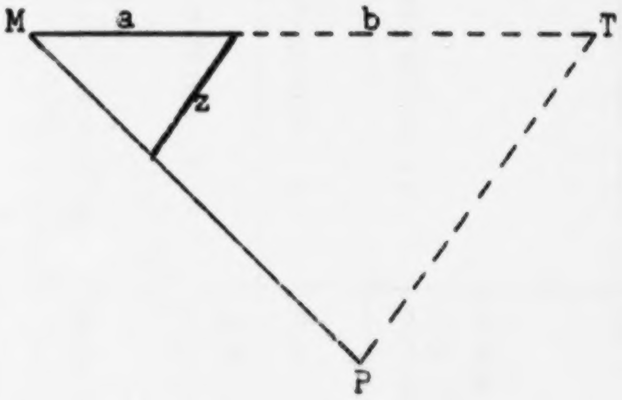
(Fig. #3)

4. In Figure #2, $\frac{y + z}{x} = \frac{b}{a}$ .... If two triangles are similar, the corresponding sides are in proportion.
5. In Figure #3, $\frac{x + y}{z} = \frac{b}{a}$ .... If two triangles are similar, the corresponding sides are in proportion.
6. From (4), $y + z = x(\frac{b}{a})$ .... Multiplication axiom
7. From (5), $x + y = z(\frac{b}{a})$ .... Multiplication axiom
8. Subtracting (5) from (4), $z - x = (x - z)(\frac{b}{a})$ Subtraction axiom
9. Multiplying (8) by (-1), $x - z = (x - z)(\frac{-b}{a})$ Multiplication axiom
10. Dividing by $(x - z)$, $1 = (\frac{-b}{a})$ .... Division axiom
11. Hence, from (10), $a = -b$; therefore, $a + b = 0$. Q.E.D.

Solution for Fallacy: The sum of the bases of a trapezoid is equal to zero.

Since $RT = NP$ and $RT \parallel NP$, RTNP is a parallelogram. RN and segments x, y, and z are all parallel to PT.

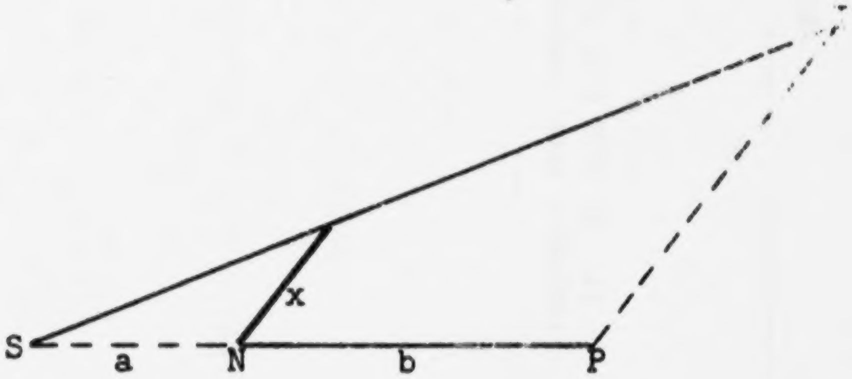
Consider $\triangle$ MPT:



Since z is parallel to PT, similar triangles are formed.

Hence $\frac{a}{b} = \frac{z}{PT}$

Now consider $\triangle$ SPT:



Since x is parallel to PT, similar triangles are formed.

Hence $\frac{a}{b} = \frac{x}{PT}$

From these two proportions, it is seen that $x = z$. Therefore, $x - z = 0$. The fallacy occurs in step (10), as division by zero is impossible.

COMPARISONS BETWEEN EUCLIDEAN AND NON-EUCLIDEAN GEOMETRIES

Non-Euclidean geometry differs from Euclidean geometry only with respect to the parallel postulate. Thus, all proofs of theorems which do not depend on the parallel postulate are the same, while those which depend on the parallel postulate differ. Some of the differences are shown below.

	Euclidean	Riemannian	Lobatshevskian
Any two lines	intersect at one point or are parallel.	intersect in one and only one point.	intersect in one and only one point, are parallel, or are non-intersecting.
Given any line m and any point P not in m, there exists	exactly one line through P and parallel to m.	no lines through P and parallel to m.	two lines through P and parallel to m.
Every line	is separated into two half-lines by a point.	is not separated into two half-lines by a point.	is separated into two half-lines by a point.
Parallel lines	are equidistant at all points.	do not exist.	converge (approach asymptotically) in one direction, diverge in the other.
If a line intersects one of two parallel lines, it	must intersect the other.	----	may or may not intersect the other.
The sum of the angles in a triangle is	equal to a straight angle.	greater than a straight angle.	less than a straight angle.
All lines perpendicular to the same line	are parallel.	intersect at a single point (pole of the line).	are non-intersecting (converge).
The area of a triangle is	unbounded, and independent of the sum of its angles.	bounded, and proportional to the excess of the sum of its angles over 180°. The greater the excess, the greater the area.	bounded, and proportional to the deficiency of the sum of its angles from 180°. The greater the deficiency, the greater the area.
Triangles with corresponding angles equal are	similar.	congruent.	congruent.
Similar triangles have the same shape and	may be of different sizes.	have the same size.	have the same size.

TOPICS AND PROJECTS FOR INDIVIDUAL ENRICHMENT IN PLANE AND SOLID GEOMETRY

Pupils should be able to do mathematics projects in plane and solid geometry. There are many reasons which can be advanced in favor of mathematics projects.

They develop independent thinking and self-reliance in the pupils.
They challenge the intellect and encourage gifted pupils to do outstanding work.
They inspire pupils to study phases of mathematics not normally covered in the regular course.

In general, projects should take the form of a model, a display or collection, an experiment, or an original piece of work. A new idea, an ingenious application of an old principle, or an unusual and attractive display of some advanced concept--all are highly desirable.

The best projects are those that grow out of the interests of the pupils. While the teacher may guide or suggest, the selection of the project should be determined by the pupil.
The teacher should insist on projects of high quality.

The criteria for judging mathematics projects, as suggested by Science Services, is as follows:

CREATIVE ABILITY - 30%	Does the work show originality in its approach? Judge originality without regard to the expense of the equipment purchased or borrowed. Give weight to clever use of material and to collections if they serve a purpose.
SCIENTIFIC THOUGHT - 30%	Is the project well organized? Is it accurate?
THOROUGHNESS - 10%	How completely is the story told? It is not essential that step-by-step details about the construction of the model be given.
SKILL - 10%	Is the workmanship good? Is the exhibit more attractive than others of the same nature?
CLARITY - 10%	Will the average person understand what is being displayed? Are all labels and other descriptions clearly presented?

ORAL PRESENTATION 10%	Is the oral presentation of the project to the class clear, well-organized, and understandable? Did the oral presentation awaken the interest of the class?

It is recommended that the teacher schedule the projects to be demonstrated over an extended period of time.
Perhaps some type of bonus arrangement can be offered to pupils willing to submit their projects early.
However, once a due date is given, it should be adhered to strictly.

The teacher should have available a wide selection of ideas and suggestions for projects. Wherever possible, source material in the form of books and pamphlets should be made available to the pupils.
Below is a list of suggested topics for pupil projects. Many of the topics are general in nature and lend themselves to specialization within a topic.
The list is of course incomplete and pupils should feel free to choose any appropriate topic for a project.
It is suggested that the teacher approve all choices of projects made by pupils before the work is begun.

References for topics and projects for pupil study will be found in the Bibliography section.

Cutting squares
A finite geometry
Three-dimensional dominoes
Duality in points and lines
Geometry of the catenary and tractrix
Desargue's theorem (both two- and three-dimensional)
Mathematics of crystals
Eratosthenes' measurement of the circumference of the earth
Model of $(a + b)^3$
Tangrams
Optical illusions
Triangle of progressions
The nine original postulates of Euclid
One and two point perspective
Geometry of bubbles and liquid film
Analytic geometry
Shells and geometric spirals
Map coloring (the four-color problem)
Topology
Time curves
Measurement of the distance to the moon by simple geometry
Primitive geometry taken from the Indians
Geometric interpretation of super perfect numbers
Cavalieri's solids
The sphereometer
Platonic solids
The sextant
The Tower of Brahma

Proof of Euler's Theorem
The geometry of knots
Feurbach's theorem
The trisection of an angle, duplication of a cube
The conic sections
Menelaus's theorem
Ceva's theorem
Geometric fallacies
Brocard points used in aviation
Non-Euclidean geometry
Geometry of the sundial
Surveying
Macheronian geometry
Halstead's rational geometry
Archimedian polyhedra
Complex polyhedra
Duality of polyhedra
Centroids
Geometric paradoxes
Hyperboloid of one sheet
Hyperbolic paraboloid
Spheroids, cylindroids, conoids, and ellipsoids
Curve stitching
Cycloids
Different proofs of the Pythagorean theorem
The golden section
The study of pi
Linkages
Geometric foundations of the theory of relativity
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Geometry in aeronautics
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Geometry in architecture
Geometry in engineering
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Geometry in automobile designing
Navigation and navigational instruments
Geometry of the slide rule
Vector geometry
Geometry of the infinite
The Fibonacci series
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Paper folding
Construction of the pyramids of Egypt
Symmetry
Symmetry in nature

Mechanical models and teaching aids
Geometric inequalities
Geometric transformations
Space geometrics
General quadric or ruled surfaces
Conics as a project
Conics as a locus
Map projections
Descriptive geometry
Lissajous' Curves

GLOSSARY

GLOSSARY

-A-

- acute
  - angle
  - triangle
- adjacent
  - angles
  - segments
- alternate
  - exterior angles
  - interior angles
- altitude
- angle
- apothem
- arc
- area
  - of a circle
  - of a parallelogram
  - of a rectangle
  - of a regular polygon
  - of a rhombus
  - of a sector
  - of a segment
  - of a square
  - of a trapezoid
  - of a triangle
- auxiliary lines
- axiom
- axis of a circle of a sphere
- axis of a cone
- axis of a cylinder

-B-

- base angles
- base
  - of an isosceles triangle
  - of a trapezoid
  - of a triangle
- bases
  - of a frustum of a pyramid
  - of a truncated pyramid
- betweenness
  - of planes
  - of points
  - of rays
- birectangular spherical triangle
- bisect
- bisector
- bisector
  - of an angle
  - of a dihedral angle
  - of a line segment
- broken line

-C-

- Cavalieri's Theorem
- center of a regular polygon
- central angles of a regular polygon
- centroid
- chord
- circle
- circular
  - cone
  - cylinder
- circumcenter
- circumference of a circle
- circumscribed
  - circle
  - polygon
- closed broken line
- collinear points
- common tangent
- complement
- complementary angles
- concave polygon
- concentric
  - circles
  - spheres
- conclusion
- concurrent lines
- cone
- cone of revolution
- congruent
  - polygons
  - triangles
- conic sections
- construction
- contrapositive
- converse
- convex
  - polygon
  - polyhedron
- coordinate
- coplanar
- corollary
- corresponding angles
- cube
- cylinder
- cylinder of revolution

-D-

- decagon
- deductive reasoning
- defined terms
- definition
- determine

diagonal
diameter
  - of a circle
  - of a sphere
dihedral angle
disjoint set
distinct points
dodecagon
dodecahedron
dome
drawing
Δ x
Δ y

-E-

element
ellipse
empty set
equal
  - arcs
  - circles
  - polygons
  - solids
  - spheres
equiangular polygon
equiangular triangle
equilateral polygon
equilateral triangle
equation
  - of a circle
  - of a line
Euler's Theorem
exterior angles
exterior of a circle
exterior of a triangle
externally tangent
extremes

-F-

face
face angle
face of a dihedral angle
finite set
frustum
  - of a cone
  - of a pyramid

-G-

geometry
great circle

-H-

half-line
half-plane
half-space
hemisphere
heptagon
heptadecagon
Hero's formula
hexagon
hexagonal prism
hexahedron
hyperbola
hypotenuse
hypothesis

-I-

icosahedron
if-then statement
implication
incenter
indirect reasoning
inductive reasoning
inequalities
  - of the same order
  - of the opposite order
infinite set
inscribed
  - angle
  - circle
  - polygon
intercepted arc
interior angles
interior
  - of an angle
  - of a circle
  - of a polygon
  - of a solid
  - of a triangle
internally tangent
intersection
  - of loci
  - of sets
inverse
isosceles
  - trapezoid
  - triangle

-L-

lateral area
  - of a cone
  - of a cylinder

- of a frustum of a cone
  - of a frustum of a pyramid
  - of a prism
  - of a pyramid

lateral edge
lateral face
legs
  - of an isosceles triangle
  - of a right triangle

length of a line segment
limit
line
line of centers
line segment
locus
lune

-M-

major arc
means
measure
  - of an angle
  - of a dihedral angle
  - of a line segment
  - of a surface

median
  - of a trapezoid
  - of a triangle

member of a set
midpoint of a line segment
minor arc

-N-

n-gon
nappes
non-Euclidean geometry
nonagon
null set

-O-

oblique
  - cone
  - cylinder
  - prism

obtuse
  - angle
  - triangle

octagon
octagonal prism
octahedron
opposite rays
origin
orthocenter

-P-

parabola
parallel
  - lines
  - planes

parallelepiped
parallelogram
pentadecagon
pentagon
perimeter
perpendicular
  - lines
  - planes

perspective drawing
pi (π)
plane
plane angle of a dihedral angle
Platonic solids
point
polar
  - distance
  - triangle

poles of a circle of a sphere
polygon
polyhedral angle
polyhedron
postulate
prism
prismatoid
projection
proportion
proportional line segments
proposition
pseudosphere
pyramid
Pythagorean Theorem

-Q-

quadrangular prism
quadrant
  - in coordinate geometry
  - of a sphere

quadrilateral

-R-

- radius
  - of a circle
  - of a regular polygon
  - of a sphere
- ratio
- ray
- rectangle
- rectangular
  - prism
  - parallelepiped
- reflex angle
- regular
  - polygon
  - polyhedron
  - prism
  - pyramid
- rhombicosidodecahedron
- rhombus
- right
  - angle
  - cone
  - cylinder
  - parallelepiped
  - prism
  - section
  - spherical triangle
- right
  - trapezoid
  - triangle
- rise
- run

-S-

- scalene triangle
- secant
- section of a solid
- sector of a circle
- segment of a circle
- semicircle
- set
- side
  - of a polygon
  - of a triangle
- similar polygons
- slant height
  - of a cone of revolution
  - of a regular pyramid
- slope of a line
- small circle
- space
- sphere
- spherical
  - angle
  - cone
  - degree
  - distance
  - excess
  - polygon
  - pyramid
  - sector
  - solid
  - triangle
  - wedge
- square
- square
  - pyramid
  - root
- straight
  - angle
  - line
- subset
- supplement
- supplementary angles
- syllogism

-T-

- tangent
- tangent
  - circles
  - spheres
- tetrahedron
- theorem
- total area
  - of a cone
  - of a cylinder
  - of a prism
  - of a pyramid
  - of a sphere
- transversal
- trapezium
- trapezoid
- triangle
- triangular
  - prism
  - pyramid
- trirectangular spherical triangle
- truncated pyramid

-U-

undefined terms
unequal arcs
union of sets
unit
  - of area
  - of volume
universal set

-V-

vertex
vertex angles
vertices of a polygon
volume of a solid

-X-

x-axis
x-coordinate
x-intercept

-Y-

y-axis
y-coordinate
y-intercept

-Z-

zone
  - of a sphere
  - of one base

BIBLIOGRAPHY

BIBLIOGRAPHY

Abbott, Edwin A. *Flatland, A Romance of Many Dimensions*. New York: Dover Publications, Inc., 1952.

Beckenbach, Edwin and Bellman, Richard. *An Introduction to Inequalities*. New York: Random House, Inc., 1961.

Bell, E. T. *Men of Mathematics*. New York: Simon and Schuster, 1937.

Bergamini, David and the Editors of *Life*. *Mathematics*. New York: Time, Inc., 1963.

Brandes, Louis Grant. *Geometry Can Be Fun*. Portland, Maine: J. Weston Walch, 1958.

Brandes, Louis Grant. *Math Can Be Fun*. Portland, Maine: J. Weston Walch, 1956.

Brumfiel, C. F., Eicholz, R. E., and Shanks, M. E. *Geometry*. Reading, Massachusetts: Addison- Wesley Publishing Company, Inc., 1960.

Bruyr, Donald L. *Mathematics Made Visual*. Columbus, Kansas: Graphic Publications, 1956.

Carroll, Lewis. *Pillow Problems and a Tangled Tale*. New York: Dover Publications, Inc., 1958.

Court, Nathan A. *Mathematics In Fun and Earnest*. New York: Dial Press, 1958.

Cundy, H. Martyn, and Rollett, A. P. *Mathematical Models*. New York: Oxford University Press, 1961.

Davis, Philip J. *The Lore of Large Numbers*. New York: Random House, Inc., 1961.

Dresden, Arnold. *An Invitation to Mathematics*. New York: Henry Holt and Company, 1936.

Eves, Howard. *An Introduction to the History of Mathematics*. New York: Holt, Rinehart, and Winston, Inc., 1960.

Gamow, George. *Mr. Tompkins in Wonderland*. New York: The Macmillan Company, 1940.

Gamow, George. *One, Two, Three . . . Infinity!* New York: The Viking Press, Inc., 1947.

Ghyka, Matila. The Geometry of Art and Life. New York: Sheed and Ward, 1946.

Glenn, William H. and Johnson, Donovan A. Adventures in Graphing. St. Louis: Webster Publishing Company, 1960.

Glenn, William H. and Johnson, Donovan A. Invitation to Mathematics. Garden City, New York: Doubleday and Company, 1962.

Glenn, William H. and Johnson, Donovan A. The Pythagorean Theorem. St. Louis: Webster Publishing Company, 1960.

Glicksman, Abraham M. Vectors In Three Dimension Geometry. Washington D. C.: National Council of Teachers of Mathematics, 1961.

Greitzer, Samuel. Elementary Topography and Map Reading. New York: McGraw-Hill Book Company, Inc., 1944.

Henderson, Kenneth B., Pingry, Robert E. and Robinson, George A. Modern Geometry, Its Structure and Function. New York: McGraw-Hill Book Company, Inc. 1962.

Hilbert, D. and Cohn-Vossen, S. Geometry and the Imagination. New York: Chelsea Publishing Company, 1952.

Hobson, E. W. Squaring the Circle. New York: Chelsea Publishing Company, 1953.

Hogben, Lancelot. Mathematics in the Making. New York: Doubleday and Company, 1960.

Hogben, Lancelot. The Wonderful World of Mathematics. Garden City, New York: Garden City Books, 1955.

Johnson, Donovan A. Curves in Space. St. Louis: Webster Publishing Company, 1963.

Johnson, Donovan A. Logic and Reasoning in Mathematics. St. Louis: Webster Publishing Company, 1963.

Johnson, Donovan A. Paper Folding for Mathematics Classes. Washington D. C.: National Council of Teachers of Mathematics, 1957.

Johnson, Donovan A. and Glenn, William H. Topology, The Rubber Sheet Geometry. St. Louis: Webster Publishing Company, 1960.

Jones, S. I. Mathematical Nuts. Nashville, Tenn: S. I. Jones Company, 1936.

Jones, S. I. Mathematical Wrinkles. Nashville, Tenn: S. I. Jones Company, 1930.

Kasner, Edward and Newman, James. Mathematics and the Imagination. New York: Simon and Schuster, Inc., 1940.

Kazarinoff, Nicholas D. Geometric Inequalities. New York: Random House, Inc., 1961.

Kline, Morris. Mathematics in Western Culture. New York: Oxford University Press, 1953.

Lieber, Lillian R. Non-Euclidean Geometry or Three Moons in Mathesis. New York: Academic Press, 1931.

Lieber, Lillian R. and Lieber, H. G. The Education of T. C. Mits. New York: W. W. Norton and Company, Inc., 1944.

Lockwood, E. H. A Book of Curves. Cambridge, England: Cambridge University Press, 1961.

Meserve, Bruce E. and Sobel, Max A. Mathematics for Secondary School Teachers. New Jersey: Prentice-Hall, Inc., 1962.

Newman, James R. The World of Mathematics. Volume I, II, III, IV. New York: Simon and Schuster, Inc., 1956.

Niven, Ivan. Numbers: Rational and Irrational. New York: Random House, Inc., 1961.

Norton, M. Scott. Basic Concepts of Vectors. St. Louis: Webster Publishing Company, 1963.

Ransom, Wm. R. One-Hundred Mathematical Curiosities. Portland, Maine: J. Weston Walch, 1955.

Ransom, Wm. R. Geometry Growing. Washington, D. C.: National Council of Teachers of Mathematics, 1954.

Ransom, Wm. R. Three Famous Geometries. Portland, Maine: J. Weston Walch, 1959.

Ravielli, Anthony. An Adventure in Geometry. New York: The Viking Press, Inc., 1957.

Commission on Mathematics. Appendices. New York: Report of the Commission on Mathematics, College Entrance Examination Board, 1959.

Sawyer, W. W. Prelude to Mathematics. Baltimore, Maryland: Penguin Books, Inc., 1956.

Schacht, John F. and McLennan, Roderick C. Brief Units in Solid Geometry. New York: Henry Holt and Company, 1959.

Schacht, John F., McLennan, Roderick C. and Griswold, Alice L. Contemporary Geometry. New York: Holt, Rinehart, and Winston, Inc., 1962.

School Mathematics Study Group. Mathematics for High School. Geometry. Parts I, II, and III. New Haven, Connecticut: Yale University Press, 1960.

Shute, William G. and Shirk, William W. and Porter, George F. Supplement to Plane Geometry. New York: American Book Company, 1959.

Smith, David Eugene. History of Mathematics. Volume II. New York: Dover Publications, Inc., 1953.

Sommerville, D. M. Y. The Elements of Non-Euclidean Geometry. New York: Dover Publications, Inc., 1958.

Steinhaus, Hugo. Mathematical Snapshots. New York: Oxford University Press, 1950.

Turney, Billy L. Geometry Teaching Aids You Can Make. Portland, Maine: J. Weston Walch, 1958.

Vergara, William C. Mathematics in Everyday Things. New York: New American Library, 1962.

Yaglom, I. M. Geometric Transformations. Syracuse, New York: L. W. Singer Company, Inc., 1962.

Yates, R. C. Geometric Tools: A Mathematical Sketch and Model Book. St. Louis: Educational Publishers, Inc., 1949.

Yates, R. C. The Trisection Problem. Ann Arbor, Michigan: Edwards Brothers, 1947.

Young, Frederick H. *Pythagorean Numbers · Congruences, A Finite Arithmetic · Geometry in the Number Plane*. Boston: Ginn and Company, 1961.

Young, Frederick H. *Random Numbers · Mathematical Induction · Geometric Numbers*. Boston: Ginn and Company, 1962.

Young, Frederick H. *The Nature of Regular Polyhedra · Infinity and Beyond · An Introduction to Groups*. Boston: Ginn and Company, 1961.